CSE 200 Homework 1 Solutions

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Read the definitions section if you are curious or don't know the definition of something used in the proofs.

1 coRE is separable

Claim 1. *Let $A, B \in$ coRE be disjoint. Then $\exists C \in$ D $(A \subseteq C \wedge B \cap C = \emptyset)$.*

Proof. Let $M_{\overline{A}}, M_{\overline{B}} \in$ TM recognize $\overline{A}, \overline{B}$. Define $M_C \in$ TM as follows.

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1  M_C(x)
2    run M_Ā(x), M_B̄(x) in parallel
3    if M_Ā(x) accepts first, then reject
4    if M_B̄(x) accepts first, then accept
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Let $C = \mathcal{L}(M_C)$.

If $x \in A$, then $x \in \overline{B} - \overline{A}$, and so $M_C(x)$ accepts. So $A \subseteq C$.

If $x \in B$, then $x \in \overline{A} - \overline{B}$, and so $M_C(x)$ rejects. So $B \cap C = \emptyset$.

If $x \in \overline{A} \cap \overline{B}$, then $M_C(x)$ halts. So $C \in$ D. □

2 RE is not separable

Claim 2. *Let $A = \{\langle M, x\rangle \mid M(x) = \text{accept}\}$, $B = \{\langle M, x\rangle \mid M(x) = \text{reject}\}$. Then $A, B \in$ RE, $A \cap B = \emptyset$, but $\neg\exists C \in$ D $(A \subseteq C \wedge B \cap C = \emptyset)$.*

Proof. $A, B \in$ RE, $A \cap B = \emptyset$ is obvious. Suppose indirectly that $C \in$ D, $A \subseteq C, B \cap C = \emptyset$. Let $M_C \in$ TM decide C. Define $N \in$ TM by

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1  N(⟨M⟩)
2    return not M_C(⟨M, ⟨M⟩⟩)
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N is a decider since M_C is a decider.

If $N(\langle N\rangle) =$ accept, then $\langle N, \langle N\rangle\rangle \notin C$, so $\langle N, \langle N\rangle\rangle \notin A$, so $N(\langle N\rangle) \neq$ accept, a contradiction. If $N(\langle N\rangle) =$ reject, then $\langle N, \langle N\rangle\rangle \in C$, so $\langle N, \langle N\rangle\rangle \notin B$, so $N(\langle N\rangle) \neq$ reject, also a contradiction.

□

3 Definitions, etc.

In the Zermelo-Fraenkel (ZF) development of set theory, a *class* is a meta-mathematical object simply referring to a formula (e.g. the class of sets not containing themselves). A class can be upgraded to a *set* if the axioms of set theory imply that it is also a set (E.g. the empty set axiom asserts the existence of a set not containing any set, so the empty class is a set.). A class is *proper* if it is not a set (E.g. Russell's paradox implies that the class of all sets is proper.). In this class, we will often use abstract models such as Turing machines which, among other things, have alphabets. An alphabet is a finite nonempty set, but the class of all such sets is proper, and so the class of Turing machines is proper as well. This phenomenon occurs in almost all abstract mathematical models (e.g. group theory, graph theory), and although it could by removed from models of objects of bounded size by insisting that all objects are constructed from some canonical set (e.g. insist that every alphabet is a set of natural numbers), it is often inconvenient. So we will have things like "the class of recursive languages", as opposed to "the set of recursive languages."

RE is the class of *recursively enumerable* languages, a.k.a. the class of *recognizable* languages. coRE $= \{\overline{A} \mid A \in \text{RE}\}$ is the class of *co-recursively enumerable* languages. D is the class of decidable languages. TM is the class of Turing machines. If $M \in$ TM, then $\mathcal{L}(M)$ is the language of M. We will use $M(x)$ sometimes to abstractly refer to the behavior of machine M on input x (e.g. "$M(x)$ does not make any queries") and sometimes as the function mapping x to one of accept, reject, loop, according to the behavior of M (e.g. "$M(x) = \text{loop}$").

Sets A, B are *disjoint* iff $A \cap B = \emptyset$. The *difference* between sets A, B is $A - B = \{x \in A \mid x \notin B\}$. The *complement* of a set A which is a subset of some implicit universe U is $\overline{A} = U - A$. If A is a language over the alphabet Σ, then implicitly $U = \Sigma^*$.

$\wedge, \vee, \neg$ mean "and", "or", "not". The quantifier symbols $\exists, \forall$ mean "there exists", and "for all". The universe over which variables are quantified is often implicit (E.g. in "$\forall c\ n^c \in O(2^n)$", the universe is either the reals, or integers, or natural numbers; the information content of the statement is the same in all 3 cases, so it really doesn't matter which one the author intended.). In this class, variables will usually be one of alphabet symbols, languages, sets of languages, reals, integers, natural numbers, real valued functions of a single variable, Turing machines. Good style demands making which one you mean clear from context or simply state a set from which the variable comes (E.g. "$\forall M \in$ TM" says "for each Turing machine M").

Wlog α means "without loss of generality, we can assume α since the case $\neg\alpha$ easily reduces to the case α". E.g. "Let M recognize the language L and wlog assume that on each input x, $M(x)$ accepts or loops."