

CSE 20: Week 5 Homework Solutions

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SF: Solutions to 1.2, 1.3, 1.6, and 1.9 are found in the back of your textbook.

Solow 1.15 A function f of one variable is *strictly increasing* if and only if for all real numbers x and y with $x < y$, $f(x) < f(y)$. Show that $f(x) = x^3$ is strictly increasing. (Hint: $\forall x, y$ where either $x \neq 0$ or $y \neq 0$, $x^2 + xy + y^2 > 0$.)

There are many ways to prove this statement. Perhaps the most concise proof is as follows:

Proof Without the loss generality, assume that $x < y$. (Note that $y - x > 0$ is positive since $x \neq y$.) From our assumption, we have $x^2 + xy + y^2 > 0$ as positive. Then, $(y - x)(x^2 + xy + y^2) > 0$ is positive since the product of two positive number is positive. Thus we have

$$\begin{aligned} 0 &< (y - x)(x^2 + xy + y^2) \\ 0 &< y^3 - x^3 \\ x^3 &< y^3 \\ f(x) &< f(y) \end{aligned}$$

Therefore $f(x)$ is strictly increasing. $\square$

One other way you might approach this proof is to consider the difference between x and y . We can construct a proof this way, but it is somewhat less concise.

Proof For all $x, y \in \mathbb{R}$, let d be the positive difference $d = y - x$ such that $d > 0$. We have

$$f(y) = f(x + d) \tag{1}$$

$$= (x + d)^3 \tag{2}$$

$$= x^3 + 3x^2d + 3xd^2 + d^3 \tag{3}$$

$$= f(x) + d[3x^2 + 3xd + d^2] \tag{4}$$

Next we show that $3x^2 + 3xd + d^2 > 0$ by substituting $e = \frac{d}{\sqrt{3}}$. Note that

$e > 0$.

$$3x^2 + 3xd + d^2 = 3x^2 + 3x(\sqrt{3}e) + (\sqrt{3}e)^2 \quad (5)$$

$$> x^2 + \sqrt{3}xe + e^2 \quad (6)$$

$$> \frac{\sqrt{3}}{2}x^2 + \sqrt{3}xe + \frac{\sqrt{3}}{2}e^2 \quad (7)$$

$$> \frac{x^2}{2} + xe + \frac{e^2}{2} \quad (8)$$

$$= \left(\frac{x}{\sqrt{2}} + \frac{e}{\sqrt{2}}\right)^2 \quad (9)$$

$$> 0 \quad (10)$$

Equation 5 follows by substitution and Equation 6 follows by algebra. Equation 7 follows since $ca^2 < a^2$ for $0 < c < 1$ since $a^2 > 0$ for $a \in \mathbb{R}$. (In our case $c = \frac{\sqrt{3}}{2}$.) Equation 8 and 9 follow from algebra and factorization. Equation 10 follow from the fact that $a^2 > 0$ for all $a \in \mathbb{R}$.

Returning to Equation 4, we have $d > 0$ and $3x^2 + 3xd + d^2 > 0$ so $d[3x^2 + 3xd + d^2] > 0$. Thus, for all $x < y$, $f(x) < f(y)$. Therefore $f(x) = x^3$ is strictly increasing. $\square$