

CSE 20: Midterm 2

February 28, 2006—Day 15

Name: _____

Student ID: _____

No books, no calculators are allowed. One page of handwritten notes are allowed. If you need to make an assumption to solve a problem, state the assumption.

1. 16 pts. Each row of the following table describes partial information about a different function. Fill in the each of the cells of the following table with *always*, *sometimes*, or *never* as to whether the function is surjective (onto) or injective (one-to-one).

Function description	Surjective	Injective
$ \text{Image}(f) = \text{Coimage}(f) $	Sometimes	Sometimes
$ \text{Image}(g) < \text{Codomain}(g) $	Never	Sometimes
$ \text{Domain}(h) < \text{Codomain}(h) $	Never	Sometimes
$ \text{Domain}(j) > \text{Codomain}(j) $	Sometimes	Never
$ \text{Codomain}(f') = 1$	Always	Sometimes
$ \text{Domain}(g') = \text{Coimage}(g') $	Sometimes	Always
$ \text{Domain}(h') > \text{Coimage}(h') $	Sometimes	Never
k is a permutation	Always	Always
$j' : \{2, 3, 4\} \rightarrow \{5, 6, 7, 8\},$ $j'^{-1}(5) = \{2\}, j'^{-1}(6) = \{3\}$	Never	Always

Solution: There were actually 18 empty spaces here, so the question was worth 18 rather than 16 points.

2. 14 pts. List all partitions of $\{a, b, c, d\}$ of cardinality 2.

Solution:

$S(4, 2) = 7$. Here are the 7 partitions:

$$\begin{aligned}
 &\{\{a, b, c\}, \{d\}\} \\
 &\{\{a, b, d\}, \{c\}\} \\
 &\{\{a, c, d\}, \{b\}\} \\
 &\{\{b, c, d\}, \{a\}\} \\
 &\{\{a, b\}, \{c, d\}\} \\
 &\{\{a, c\}, \{b, d\}\} \\
 &\{\{a, d\}, \{b, c\}\}
 \end{aligned}$$

3. 14 pts. Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. For this exercise, answer each part with an actual number as well as showing your work.

- (a) How many functions are there with domain A , codomain B and whose image has cardinality 2?

Solution: We need for the coimage a partition of A of size 2, to choose 2 elements from B for the image, and then to establish mappings between the coimage and the image: $S(|A|, 2) \cdot \binom{|B|}{2} \cdot 2! = 7 \cdot 3 \cdot 2 = 42$.

- (b) How many non-functional relations are there on $A \times B$?

Solution: A relation is a subset of the cross-product, which has $|A| \cdot |B| (= 4 \cdot 3 = 12)$ elements. The number of subsets of that set is the size of the power set, which is 2^{12} . There are $|B|^{|A|} = (3^4 = 81)$ functional relations from B to A . The difference is $4096 - 81 = 4015$.

4. 8 pts. We intend to prove that, for all integers $k \geq 1$,

$$\sqrt{k} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}.$$

It is clearly true for $k = 1$. Assume the Induction Hypothesis (IH) that $\sqrt{n} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$. What is a correct way of concluding this proof by induction?

- (a) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n}} \geq \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{n}} \geq \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$.
- (b) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}} \geq \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$.
- (c) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \sqrt{n+1} + 1 \geq \sqrt{n+1}$.
- (d) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}$.
- (e) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + 1 \geq \sqrt{n+1}$.

Solution: B is correct

5. 20 pts. Prove by induction that $5^n - 1$ is divisible by 4, for each integer $n \geq 0$.

Solution: Base case: when n is 0, $5^0 - 1 = 1 - 1 = 0$, which is divisible by 4.

Inductive step: inductive hypothesis: $5^n - 1$ is divisible by 4, for $n \geq 0$.

To show: $5^{n+1} - 1$ is divisible by 4.

$5^{n+1} - 1 = 4 \cdot 5^n + 1 \cdot 5^n - 1$. By the inductive hypothesis, $5^n - 1$ is divisible by 4. But clearly $4 \cdot 5^n$ is also divisible by 4. Thus, we've shown that $5^{n+1} - 1$ is divisible by 4.

Alternatively, $5^{n+1} - 1 = 5 \cdot 5^n - 1$. By the inductive hypothesis, $5^n - 1$ is divisible by 4, so $5^n = 1 \pmod{4}$. $5 = 1 \pmod{4}$, so we have $5 \cdot 5^n - 1 = 1 \cdot 1 - 1 = 0 \pmod{4}$. Thus, $5^{n+1} - 1$ is divisible by 4.

6. 15 pts. Prove that $((A \cup B) \cap (C \cup D)) - D \subseteq (B - D) \cup (A - D)$.

Solution: Suppose x is in the left-hand-side. Then x is in A , or x is in B . In addition, x is in C or x is in D . Finally, in addition to all that, x is not in D . Therefore, x is in A or x is in B , and x is in C and x is not in D .

Thus, we can conclude (by ignoring the restriction on C) either:

- x is in B and x is not in D
- or, x is in A and x is not in D .

Thus, $x \in B - D$, or $x \in A - D$. Combining, $x \in (B - D) \cup (A - D)$. Thus, we've shown that if x is a member of the LHS, x is a member of the RHS. This shows that the LHS is a subset of the RHS.

7. 13 pts. Let f be the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 1 & 7 & 5 & 4 & 8 & 6 & 3 & 2 \end{pmatrix}$$

- (a) Write f^{-1} in cycle form.

Solution: $f = (1, 9, 2)(4, 5)(6, 8, 3, 7)$. $f^{-1} = (2, 9, 1)(5, 4)(7, 3, 8, 6)$.

- (b) Write f^{98} in cycle form.

Solution: For a cycle of length k , f^{98} is equivalent to $f^{98 \bmod k}$. Thus,

$$f^{98}(1) = f^2(1) = 2$$

$$f^{98}(2) = f^2(2) = 9$$

$$f^{98}(9) = f^2(9) = 1$$

$$f^{98}(4) = f^0(4) = 4$$

$$f^{98}(5) = f^0(5) = 5$$

$$f^{98}(6) = f^2(6) = 3$$

$$f^{98}(8) = f^2(8) = 7$$

$$f^{98}(3) = f^2(3) = 6$$

$$f^{98}(7) = f^2(7) = 8$$

So, $f^{98} = (2, 9, 1)(4)(5)(6, 3)(7, 8)$.

8. 20 pts. Extra credit: Prove that if m and n are any positive integers and m is odd, then: $\sum_{k=0}^{m-1} (n+k)$ is divisible by m .

Solution:

$$\begin{aligned} \sum_{k=0}^{m-1} (n+k) &= \\ \sum_{k=0}^{m-1} n + \sum_{k=0}^{m-1} k &= \\ n \sum_{k=0}^{m-1} 1 + \frac{(m-1)m}{2} &= \\ nm + \frac{(m-1)m}{2} \end{aligned}$$

Clearly, nm is divisible by m . If m is odd, then $m-1$ is even. Let l be an integer such that $2l = (m-1)$. The second term is equal to $2lm/2 = lm$ which is divisible by m . Since both terms are divisible by m , the sum is divisible by m .