CSE 20—Discrete Math

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Predicate Logic

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Predicates

Grammatical origin

Alternate definitions of a predicate:

- A sentence with a finite number of variables. Replacing variables with specific values yields a *statement.* The domain is the set of all possible values to substitute.
- A function whose codomain is (true/false) statements.

Example:

- "$x^2 > 1$"
  - Domain?
  - Example statements?
- "Boy x likes girl y" (alternate syntax:)
  - Domain?
  - Example statements?

Important Sets

Z (*Zahl* = Number)

Z^+

N (*Natural*)

- No general agreement on whether it includes 0

Q (*Quotient)*

R (*Real)*

P (*Prime*)

Truth Set of a Predicate, P(x)

The set of all elements in the domain that make P(x) true

Denoted:

- $\{x \in D: P(x)\}$

Example:

- What is the truth set of "x is a factor of 8"?
  - If D is {0, 1, 2}?
  - if D is in {0, 1, 2, 3, …} (**N**)?
  - If D is in {…, -3, -2, -1, 0, 1, 2, 3, …} (**Z**)?

Quantifiers

Universal
- $\forall x \in D\ S(x)$

Existential
- $\exists x \in D: S(x)$

Common to drop $\in D$ if it is clear from context

Examples

Let L(x, y) be the predicate "Boy x likes girl y"

- $\exists y: \exists x: L(x, y)$
- $\exists y: \forall x\ L(x, y)$
- $\forall x\ \exists y: L(x, y)$
- $\forall y\ L(\text{"Jose"}, y)$

Negation and Quantifiers

$\sim(\exists x: P(x)) \leftrightarrow \forall x \sim P(x)$

$\sim(\forall x\ P(x)) \leftrightarrow \exists x: \sim P(x)$

Examples

L(x, y) = "x likes y" Domain: all people in my son's 1st grade class
- Everyone likes Jill
- Nobody likes Jill
- Not everybody likes Jill
- There exists a person that nobody likes
- Nobody dislikes everybody
- Everybody likes him/herself
- $\forall x\ \forall y\ \forall z\ (L(x, y) \wedge L(y, z) \rightarrow L(x, z))$
- $\forall x\ \forall y\ (L(x, y) \rightarrow L(y, x))$

Converting to Predicates

If a number is a natural number, it is an integer

The square of any real number greater than 2 is greater than 4

Quantifiers and Adversaries

Statements involving quantifiers can be viewed as a game

- Two players
  - You, trying to prove the statement is true
  - An adversary, trying to prove the statement is false
- Work from left to right through the quantifiers
  - ∀y: the adversary selects an y (think of the adversary as choosing the worst possible y). That y remains *bound* the rest of the game
  - ∃x: you select an x that will satisfy the statement. That x remains *bound* the rest of the game

Example:

- ∀x∈Z, ∃y∈Z: 2x = y
- ∃x∈Z: ∀y∈Z, xy = 0
- ∀y∈Z, ∃x∈Q: x≠0→xy = 1

Limit

The limit of a sequence *a* is *L* means that a gets arbitrarily close to L

$$\lim_{n \longrightarrow \infty} a_n = L$$

$$\forall \varepsilon \in R^+, \exists n_0 \in N : \forall n \in N, n \geq n_0 \rightarrow L - \varepsilon \leq a_n \leq L + \varepsilon$$

Order of Growth of a Function

We say that T(n) has an asymptotic upper bound of f(n) if:

$$\exists n_0 \in N : \exists c \in R^+ : \forall n \in Z, n \geq n_0 \rightarrow 0 \leq T(n) \leq cf(n)$$

Proving Quantified Statements

Prove universal statement: $\forall x \in D, P(x) \rightarrow Q(x)$

- Exhaustive enumeration

- Generalizing from the generic particular
  - "Suppose x is in D and P(x)"
  - …
  - Therefore Q(x)
- Example: The difference of two odd numbers is even

Proving Quantified Statements

Prove existential statement: $\exists x \in D: P(x)$

- Constructive proof
  - Display an x
  - Give a set of directions for finding x
- Nonconstructive proof
  - Proof by contradiction (assume non-existence and show a contradiction)
  - Show x must exist
- Example:
  - In a group of 367 people, at least two share a birthday

Disproving Quantified Statements

Disprove universal statement: $\forall x \in D, P(x) \rightarrow Q(x)$

- Counterexample
  - Show an x in D where P(x), but not Q(x)
- Example: All primes are of the form $2^n - 1$

Disprove existential statement: $\exists x \in D: P(x)$

- Equivalent to:
  - Prove
  - Or, alternatively,

  - Therefore, best bet is generalizing from the generic particular.
- Example: There exists a prime which can be written as the square of an integer > 1