CSE 20—Discrete Math

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Propositional Logic

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Simplifying a Boolean Function

Example
- (~ (~pq))(p+q)

Logic

Propositional Logic
- Deals with individual statements that have true/false values
  - no in-between
- Corresponds 1-1 with Boolean Functions and Combinational Circuits

(First-order) Predicate Logic
- Can actually talk about groups: For all, There exists,
- Covered next lecture

Propositional Logic Formulas

Well-formed formulas (wff)
- Any lower-case letter is a wff (atomic formula)
  - p
  - q
  - r
  - …
- If A is a wff, then ~A is a wff
- If A and B are wffs, then:
  - (A ∨ B) is a wff
  - (A ∧ B) is a wff
  - (A →B) is a wff
  - (A ↔ B) is a wff

As a shortcut, we sometimes omit unneeded parentheses

Formula

An atomic formula (p) represents some statement that has a truth value

- (p ∨ ~p) is true
  - *tertium non datur*
- The statement is either true or false (bivalent)

Example English propositions

- It will rain tomorrow
- The sky is blue
- The sky is fluorescent orange
- The program will finish in less than 3 minutes

The truth value of a complex formula can be determined from the truth values of the atomic formulas within

Equivalent Formulas

Formulas that differ syntactically are said to have different *forms*

- Formulas with different forms are *equivalent* if their truth tables are the same
- Example:

Formulas Equivalent to a Constant

A formula whose truth tables values are all true is called a *tautology*

- Linguistic origin:
- True regardless of the values of the atomic formulas
- Example:

A formula whose truth table values are all false is called a *contradiction*

- Linguistic origin:
- False regardless of the values of the atomic formulas
- Example:

Implication

Implication (also called if-then)

- p → q
- For the formula to be true, if p is true, then q can't be false.
- If p is false, the formula is true

p	q	p →q
f	f	
f	t	
t	f	
t	t	

- Example:
  - If it is raining tomorrow, the driveway will get wet

Implication

Common English forms

- *If* p *then* q
  - If it is raining, then the driveway will get wet
- p *only if* q
  - The team will play in the championship game *only if* the team reaches the semifinals
- p *if and only if* q
  p *iff* q
  - The team will be the champion *if and only if* the team wins the championship game
- p *is necessary for* q
  A necessary condition for q *is* p
  - Reaching the semifinals *is necessary for* playing in the championship game

Implication

Common English forms

- p *is sufficient for* q
  A sufficient condition for q *is* p
  - "John is eligible to vote" *is sufficient (to ensure the truth of)* "John is at least 18 years old"
- p *is necessary and sufficient for* q
  - "X is even" *is necessary and sufficient (to ensure the truth of)* "X is evenly divisible by 2"

Relatives of Implication

Given p→q

- Inverse: ~p→~q
- Converse: q→p
- Contrapositive: ~q→~p

p	q	p→q	~p→~q	q→p	~q→~p
f	f	t			
f	t	t			
t	f	f			
t	t	t			

Anything can be Proved from a Contradiction

If we can establish:

- A contradiction implies anything because:
  - contradiction → q
  - contradiction
  - Therefore, q

Proof by Contradiction

To prove q by contradiction
- Assume ~q
- Show this leads to a contradiction

- Example
  - Prove that there is no largest integer (p)
  - Assume, to the contrary that _____ (~p)
  - Then, _____ (formula that follows from p)
  - Now, _____ ($p \wedge \sim p$)

Argument forms

Classical logic uses syllogisms
- Two premises
- One conclusion

- *Modus ponens*
  - If it rained, the driveway got wet
  - It rained
  - Therefore, the driveway got wet

- *Modus tollens*
  - If it rained, the driveway got wet
  - The driveway didn't get wet
  - Therefore, it didn't rain

Argument Forms

Generalization
- p
- Therefore, $p \vee q$

Specialization
- $p \wedge q$
- Therefore, p

Conjunction
- p
- q
- Therefore, $p \wedge q$

Elimination
- $p \vee q$
- ~q
- Therefore, p

Transitivity
- $p \rightarrow q$
- $q \rightarrow r$
- Therefore, $p \rightarrow r$

Argument Forms

Proof by Division into Cases
- $p \vee q$
- $p \rightarrow r$
- $q \rightarrow r$
- Therefore, r

Contradiction rule
- $\sim p \rightarrow$ false
- Therefore, p

Algebraic Rules for Propositional Formulas

Equivalences between propositional formulas (similar to algebraic equivalences):

- Associative
- Distributive
- Idempotent
- Double negation
- DeMorgan's
- Commutative
- Absorption
- Bound
- Negation