

CSE 20—Discrete Math

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Order

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Ordered Subset

If $(S, \leq)$ is a poset, then,

- Take $X \subseteq S$
- Take $\leq_X = \{a \leq b : a, b \in X\}$

$(X, \leq_X)$ is a poset

We would normally write: X is an **ordered subset** of S

- Assume the order relation of X is the relation induced by the order relation of S

Example: given $(\mathbb{Z}, |)$ take $\{1, 2, 3, 4, 5, 6, 7, 8\} \subseteq \mathbb{Z}$

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Product Set

Given two posets $(S, \leq_S)$, and $(T, \leq_T)$, we define the product set $(R, \leq_R)$

- $R = S \times T$
- $(s, t) \leq_R (s', t')$ iff $s \leq_S s'$ and $t \leq_T t'$

The product set is a poset

Example

- $(\mathbb{N} \times \mathbb{N}, \leq)$

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Linearly Ordered Sets

If $(S, \leq)$ is a poset, then a and b in S are **comparable** iff

- $a \leq b$ or $b \leq a$

If $(S, \leq)$ is a poset, and each pair of elements of S are comparable, then S is a **linearly ordered set** (or **totally ordered set**)

- $\leq$ is a **total ordering** of S
- S is called a **chain**

Examples

- $(\mathbb{N}, \leq)$
- $(\mathbb{N}, |)$
- $(\{1, 2, 32, 8, 4\}, |)$

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Maximal, Minimal, Least, Greatest

Given a poset $(S, \preceq)$, and an element e in S

- e is **minimal** if there is no preceding element of S (other than e)
- e is **maximal** if e precedes no element of S (other than e)
- e is **least** if it precedes all elements of S
- e is **greatest** if it is preceded by all elements of S

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Upper and lower bounds

Given a poset $(S, \preceq)$ and a subset of elements $A \subseteq S$, $x \in S$ is:

- an **upper bound** of A if for all $y \in A$, $y \preceq x$
 - Only some subsets have upper bounds
- a **lower bound** of A if for all $y \in A$, $x \preceq y$
 - Only some subsets have lower bounds
- a **least upper bound** of A if it is an upper bound of A and less than all other upper bounds of A
- a **greater lower bound** of A if it is a lower bound of A and greater than all other lower bounds of A

Example:

- $\{\{3, 7, 14, 21, 40, 42\}\}$

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Lattice

A poset $(S, \preceq)$ is a **lattice** if for pair of elements in S has a least upper bound (*join*) and a greatest lower bound (*meet*)

- Hasse diagram looks like a physical lattice

Example:

- $\{\mathbf{N}, \leq\}$

- $\{S, \subseteq\}$

- $\{\mathbf{Z}^+, |\}\}$

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Topological Sort of a Poset

Given a poset $(S, \preceq)$, it's always possible to enumerate the elements of S in a linear ordering $s_1, s_2, s_3, \dots, s_{|S|}$. That linear ordering is called a **topological sort** of the poset

- $s_i \preceq s_j \rightarrow i < j$

Is the topological sort unique?

Constructing a topological sort from a Hasse diagram

```
Topological-sort(S)
  if S is empty then
    return S
  else
    e = a minimal element of S
    return Concat(e, Topological-sort(S - e))
```

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Example Topological Sort

Topological sort of prerequisites

- 8A
- 8B: requires 8A
- 12: requires 8B,
- 20: requires 12
- 21: requires 20
- 30: requires 12, 20
- 100: requires 12, 21
- 101: requires 12, 21, 100
- 105: requires 12, 21
- 120: requires 100, 101, 141
- 131A: requires 30, 100, 105
- 131B: requires 30, 100, 105, 131A
- 140: requires 20, 30
- 141: requires 140