

CSE 20—Discrete Math

Winter, 2006

March 7, Day 17
Series

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Sequence Cagematch

Given two sequences a_n and b_n , which grows faster?

- Polynomial of degree n versus polynomial of degree $n-1$?
- Exponential versus polynomial?
- Polynomial versus logarithmic?

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Identifying a Sequence

The Online Encyclopedia of Integer Sequences

- <http://www.research.att.com/~njas/sequences>

Enter part of a sequence:

- 0 1 3 6 10 15 21

Get back a list of known related sequences

ID Number: [A000217](#) (Formerly M2535 and N1002)
URL: <http://www.research.att.com/projects/OEIS?Anum=A000217>
Sequence: 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1326, 1378, 1431
Name: Triangular numbers: $a(n) = C(n+1, 2) = n(n+1)/2 = 0+1+2+\dots+n$.
Comments: Number of edges in complete graph of order n , K_n .
Number of legal ways to insert a pair of parentheses in a string of n letters. E.g. there are 6 ways for three letters: (a)bc, (ab)c, (abc), a(b)c, a(bc), ab(c). [Proof: there are $C(n+2, 2)$ ways to choose where the parentheses might go, but $n+1$ of them are illegal because the parentheses are adjacent.] Cf. [A002415](#).
For $n \geq 1$ $a(n) = n(n+1)/2$ is also the genus of a nonsingular curve of degree $n+2$ like the Fermat curve $x^{n+2} + y^{n+2} = 1$ - Ahmed Fares (ahmedfares(AT)my-deja.com), Feb 21 2001
 $a(n)$ is the number of ways in which $n+2$ can be written as a sum of three positive integers if representations differing in the order of the terms are considered to be different. In other words $a(n)$ is the number of positive integral solutions of the equation $x + y + z = n+2$. - Amarnath Murthy (amarnath_murthy(AT)yahoo.com), Apr 22 2001

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Infinite Series

Our approach will *not* be based on Bender. You are not responsible for IS section 3

- But, you are responsible for this lecture!

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Paradoxes with Infinite Sums

$$S = 1 + 1/2 + 1/4 + 1/8 + \dots$$

$$S = \underline{\hspace{2cm}}$$

$$T = 1 + 2 + 4 + 8 + 16 + \dots$$

$$T = \underline{\hspace{2cm}}$$

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Bound of $\sum_{k \in K} a_k$

Assume each a_k is non-negative

K may be infinite

If there exists a constant A such that, for all *finite* subsets $F \subset K$:

$$\sum_{k \in F} a_k \leq A$$

- Then, we say $\sum_{k \in K} a_k$ converges to the least such A , $A_{\min}$
- If there is no such A , then we say the sum *diverges* to infinity.

If K is the set of non-negative integers:

$$\sum_{k \in K} a_k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k$$

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Examples

$$\sum_{k \geq 0} x^k = \lim_{n \rightarrow \infty} \frac{1 - x^{n+1}}{1 - x}$$

- if $0 \leq x < 1$:
- if $x \geq 1$:

$$S = 1 + 1/2 + 1/4 + 1/8 + \dots =$$

$$T = 1 + 2 + 4 + 8 + 16 + \dots =$$

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Negative Terms in Infinite Series

Three different answers

$$\sum_{k \geq 0} (-1)^k$$

$$\bullet 1 + -1 + 1 + -1 + \dots$$

$$\bullet 1 + -1 + 1 + -1 + \dots$$

- Use formula for Geometric series

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Negative Terms in Infinite Series

Two different answers

$$\sum_{k \geq 0} \frac{1}{k+1} + \sum_{k < 0} \frac{1}{k-1}$$

$$\cdots + -\frac{1}{4} + -\frac{1}{3} + -\frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- Sum of n innermost parentheses =

$$\cdots + -\frac{1}{4} + -\frac{1}{3} + -\frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- Sum of n innermost parentheses =

$$\cdots + -\frac{1}{4} + -\frac{1}{3} + -\frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- Sum of n innermost parentheses =

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Infinite Sums with Positive and Negative Terms

Break apart sums into positive terms and negative terms

- Define x^+ and x^- :
 - If $x \geq 0$, $x^+ = x$, $x^- = 0$
 - If $x < 0$: $x^+ = 0$, $x^- = -x$

$$\sum_{k \in K} a_k = \sum_{k \in K} a_k^+ - \sum_{k \in K} a_k^-$$

Let $A^+ =$ $A^- =$

If A^+ and A^- are finite

- Sum converges **absolutely** to $A^+ - A^-$

If A^+ is infinite, but A^- is finite:

- Sum diverges to infinity

If A^+ is finite, but A^- is infinite:

- Sum diverges to negative infinity

If A^+ and A^- are infinite

- Sum is undefined

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Conditional Convergence

A sum where $\sum_{k \geq 0} a_n$ converges, but $\sum_{k \geq 0} |a_n|$ doesn't

- Example (alternating harmonic series)
 - $1 - 1/2 + 1/3 - 1/4 + 1/5 - \dots = \ln 2$
- Kind of wacky, because rearranging the order of terms in the series can yield different values to which the series converges
 - In fact, can rearrange the order to obtain any desired value!
 - $(1 - 1/2) - 1/4 + (1/3 - 1/6) - 1/8 + (1/5 - 1/10) - \dots = (1/2)\ln n$

Advantages of absolute convergence

- Can continue to use distributive, commutative, and associate laws of sums

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Examples

$$\sum_{k \geq 0} (-1)^k$$

$$\sum_{k \geq 0} \frac{1}{k+1} + \sum_{k < 0} \frac{1}{k-1}$$

$$\sum_{k \geq 0} \frac{-1}{2^{\frac{k+1}{2}}} [k \text{ is odd}] + \frac{1}{5^{\frac{k}{2}}} [k \text{ is even}]$$

$$\sum_{k \geq 0} \frac{-1}{n} [k \text{ is odd}] + \frac{1}{5^k} [k \text{ is even}]$$

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Harmonic Series

$1 + 1/2 + 1/3 + 1/4 + 1/5 \dots$

- Lower and upper bounds on:

$$H_n = \sum_{1 \leq k \leq n} \frac{1}{k}$$

- Put one term in group 1, two in group 2, four in group 3, etc.
- $1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10 + \dots$
- Each group has terms no bigger than first term and strictly bigger than first term of next group.
- $\frac{1}{2^{k-1}} < \text{sum of each group} \leq \frac{1}{2^{k-2}}$
- If n is in group k , $\frac{1}{2^{k-1}} < H_n \leq \frac{1}{2^{k-2}}$

$$\frac{\lfloor \lg n \rfloor}{2} < H_n \leq \lfloor \lg n \rfloor + 1$$

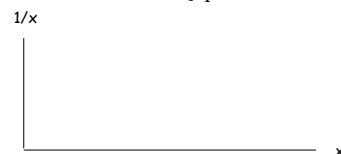
Gives bound to H_n within a factor of 2

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Better Bounds for Harmonic Series

Use integration of $f(x) = 1/x$ $\int_1^n \frac{1}{x} dx = \ln n$

- Lower bound:



- $H_n > \ln n$

- Upper bound:



- $H_n < \ln n + 1$

Bounds to H_n within at most 1

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