

CSE 20—Discrete Math

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Induction (cont.)
Sequences

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Sometimes Easier to Prove Stronger Result

In trying to devise a proof by mathematical induction, you may fail for two opposite reasons.

- You may fail because you try to prove too much: Your $P(n)$ is too heavy a burden.
- Yet, you may also fail because you try to prove too little: Your $P(n)$ is too weak a support.

In general, you have to balance the statement of your theorem so that the support is just enough for the burden.

—Polya

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Example of Strengthening the Statement

Given:

- $K_0 = 1$
 $K_{n+1} = 1 + \min(2K_{\lfloor \frac{n}{2} \rfloor}, 3K_{\lfloor \frac{n}{3} \rfloor})$

Show $K_n \geq n$

- Base case:
- Inductive step:
 - Inductive hypothesis
 - To show:

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Example

In a round-robin tournament, show that the n teams can be labeled $T_1, T_2, \dots, T_n$ such that T_i beats T_{i+1} for all $1 \leq i \leq n-1$.

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Induction and Algorithms

Algorithm correctness

- Precondition: predicates describing the initial state (involving the input variables)
- Postcondition: predicates describing the final state (involving the input and output variables)

Example:

- `Product(int m, double r)` // precondition: m a non-negative integer, r : a real
// postcondition: returns $m * r$
- `Sort(int[] y)` // precondition: y is an array of integers
// postcondition: y is a permutation of the original array such that $y[i] \leq y[i+1]$ for $0 \leq i < y.length$

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Loop Invariant

Format:

- //Loop precondition:
 Loop
 // Loop invariant:
 exit when ExitCondition
 Loop Contents
 End
 // Loop postcondition:

To Prove:

- loop precondition $\rightarrow$ Loop invariant
- Loop Invariant (at one iteration) $\wedge$ Loop contents $\wedge$!ExitCondition $\rightarrow$ Loop Invariant (at next iteration)
- ExitCondition is eventually true
- Loop invariant $\wedge$ ExitCondition $\rightarrow$ Loop postcondition

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Loop Invariant

```
double total = 0;
int i = 0;
// loop precondition: i = 0, m: non-negative integer: r is real, total=0
loop
  // loop invariant: total = i * r
  exit when i = m
  total = total + r;
  i = i + 1;
end
// loop postcondition: total = m * r
```

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Infinite Sequences

An infinite sequence of real numbers:

- consists of terms: $a_0, a_1, a_2, \dots$
 - or, may start after 0:
 $a_j, a_{j+1}, \dots$
- is a function $f: \mathbf{N} \rightarrow \mathbf{R}$
 - or, may start after 0:
 $f: (\mathbf{N} + n_0) \rightarrow \mathbf{R}$
- are normally written with subscripts rather than functional notation
- can be a **tail** of another sequence:
 - $a_2, a_3, a_4, \dots$ is a tail of $a_1, a_2, a_3, \dots$

Examples:

- $1/n$ for $n \geq 1$
 - What is the function?
 - Domain?
- $2n+1$ for $n \geq 3$
 - What is the function?
 - Domain?
- $1/5, 1/6, 1/7, 1/8, \dots$
 - What is the function?
 - Domain?
- $1/n$
 - What is the function?
 - Domain?

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Limits

Given an infinite sequence a_n :

- $a_0, a_1, a_2, \dots, a_n, \dots$

A is the limit of a_n (as n approaches infinity),

- that is, $\lim_{n \rightarrow \infty} a_n = A$

if, for all epsilon, there's a tail of the sequence a_n whose values are all less than epsilon away from A .

$$\forall \epsilon > 0, \exists N_\epsilon : \forall n \geq N_\epsilon |A - a_n| \leq \epsilon$$

If such a limit exists, we say that a_n **converges** to A .

If not, we say that a_n **diverges**.

- Could diverge to ∞
- Could diverge to $-\infty$
- Could diverge to neither

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Examples

0, 1, 2, 3, ...

1, 1, 1, 1, 1, 1, 1, ...

3, 3.1, 3.14, 3.141, 3.1415, ..., π to a million places, 3.14, 3.14, ...

1, 1/2, 1/4, 1/8, 1/16, 1/32, ...

1, -1, 10, -10, 100, -100, 1000, -1000, ...

-10, -20, -30, -40, ...

$-1^n/n$

-1^k

$\sin(x)$

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Manipulating Limits

If a_n and b_n are sequences with limits A and B , then:

- limit of $a_n b_n = AB$
- limit of $Ca_n + Db_n = CA + DB$ *linear combination*
- limit of $a_n/b_n = A/B$ *as long as $b_n \neq 0$ for large enough n*

Example

- limit of $(3n+5)/(4n+3)$

- limit of $(\log n)/n$

- limit of $n/(n+1)$

- limit of $(\sin n)/n$

L'Hospital's rule: if limit of $f(x)$ and $g(x)$ are both 0 or both ∞ , then limit of $f(x)/g(x) = \text{limit of } f'(x)/g'(x)$ (if such a limit exists)

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Boundedness

A sequence a_n is **bounded** if there is a positive number B such that for all elements, a_i , of a_n : $|a_i| < B$.

Examples

- $f(x) = \sin(x)$
- 1, 1/2, 1/4, 1/8, ...

Convergent sequences are always bounded

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Monotonicity

A sequence a_n is

- **strictly increasing** if each term is bigger than the previous one
- **strictly decreasing** if each term is smaller than the previous one
- **nondecreasing** (or weakly increasing) if each term is no smaller than the previous one
- **nonincreasing** (or weakly decreasing) if each term is no bigger than the previous one
- **monotonic** (or monotone) if it is either nondecreasing or nonincreasing

A sequence is **eventually [...]** if some tail of the sequence is [...]

Examples

- 1, 3, 5, ...
- $x^2 - 8x + 1$:

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Bounded and Monotone

If a sequence is bounded and eventually monotone, then it converges.

Example:

- limit of $2 + 1/n$
- limit of $2 - 1/n$

Sandwich theorem:

- If a_n has a limit of A and c_n has the same limit, and each term of b_n has the property that:
 - $a_n \leq b_n \leq c_n$
- Then, the limit of b_n is A

Example:

- limit of $2 + (-1)^n/n$

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Sequence Cagematch

Given two sequences a_n and b_n , which grows faster?

- Polynomial of degree n versus polynomial of degree $n-1$?
- Exponential versus polynomial?
- Polynomial versus logarithmic?

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Identifying a Sequence

The Online Encyclopedia of Integer Sequences

- <http://www.research.att.com/~njas/sequences/index.html>

Enter part of a sequence:

- 0 1 3 6 10 15 21

Get back a list of known related sequences

```
ID Number: A000217 (Formerly M2535 and N1002)
URL: http://www.research.att.com/projects/OEIS?Anum=A000217
Sequence: 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171,
190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561,
595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128,
1176, 1225, 1275, 1326, 1378, 1431
Name: Triangular numbers: a(n) = C(n+1, 2) = n(n+1)/2 = 0+1+2+...+n.
Comments: Number of edges in complete graph of order n, K_n.
Number of legal ways to insert a pair of parentheses in a string of n
letters. E.g. there are 6 ways for three letters: (a)bc, (ab)c, (abc),
a(b)c, a(bc), ab(c). [Proof: there are C(n+2, 2) ways to choose where
the parentheses might go, but n+1 of them are illegal because the
parentheses are adjacent.] Cf. A002415.
For n >= 1 a(n)=n(n+1)/2 is also the genus of a nonsingular curve of
degree n+2 like the Fermat curve x^(n+2) + y^(n+2) = 1 - Ahmed Fares
(ahmedfares(AT)my_deja.com), Feb 21 2001
a(n) is the number of ways in which n+2 can be written as a sum of three
positive integers if representations differing in the order of the
terms are considered to be different. In other words a(n) is the number
of positive integral solutions of the equation x + y + z = n+2. - Amarnath
Murthy (amarnath_murthy(AT)yahoo.com), Apr 22 2001
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