

CSE 20—Discrete Math

Winter, 2006

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Functions

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Types of Functions

Surjection (onto): Every value in the codomain is taken on at **least once**

$$\forall b \in B, \exists a \in A: f(a) = b$$

Injection (one-to-one): Every value in the codomain is taken on at **most once**.

$$\forall a \in A, b \in B, f(a) = f(b) \rightarrow a = b$$

Bijection (one-to-one and onto): Every value in the codomain is taken on **exactly once**.

$$\forall b \in B, \exists \text{ unique } a \in A: f(a) = b$$

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Example

Let

- S be the set of students attending UCSD
- I be the set of student ID numbers for those students
- D be the set of dates (MM/DD/YYYY) for the past 100 years
- G be the set of possible grade point averages between 2.0 and 3.5 (rounded to the nearest tenth)

Is the following an injection, bijection, or surjection, or none?

- The domain is S, the codomain is I and the function maps each student to his/her ID number
- The domain is S, the codomain is D and the function maps each student to his or her birthday
- The domain is D, the codomain is I and the function maps each date to the ID number of a student born on that date. If more than one, the lexicographically least ID number is chosen.
- The domain is S, the codomain is G and the function maps each student to his or her GPA rounded to the nearest tenth.
- The domain is G, the codomain is I and the function maps each GPA to the ID number of a student with that GPA. If there is more than one, the lexicographically least ID number is chosen.

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Example

Bijjective, injective, surjective, or neither?

- $f: \mathbf{R} \rightarrow \mathbf{R} \quad f(x) = x$
- $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+ \quad f(x) = x^2$
- $f: \mathbf{R} - \{0\} \rightarrow \mathbf{R}^+ \quad f(x) = x^2$
- $f: \mathbf{R} \rightarrow \mathbf{R} \quad f(x) = x^2$
- $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+ \quad f(x) = \sqrt{x}$
- $f: \mathbf{R} \rightarrow \mathbf{R} \quad f(x) = (x-1)x(x+1)$
- $f: \mathbf{R} \rightarrow \mathbf{R} \quad f(x) = e^x$
- $f: \mathbf{R} \rightarrow \mathbf{R}^+ \quad f(x) = e^x$

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Bijections

Given two sets A and B

- $|A| = |B|$ iff there exists a bijection $f: A \rightarrow B$

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Inverse Functions

If $f: A \rightarrow B$ is a bijection, then f^{-1} is the **inverse** of f . $f^{-1}: B \rightarrow A$

- if $f(a)=b$, then $f^{-1}(b) = a$

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Numbers of Functions

How many functions of the form $f: A \rightarrow B$?

- Equivalently, what is $|B^A|$?
- How many choices for first element of A?
- How many choices for second element of A?
- ...
- How many choices for last element of A?

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Relations

If A and B are sets, a subset, R, of $A \times B$ is a **relation** from A to B.

- Given relation R, if, for all a in A, there is a unique b in B such that (a, b) in R, then R is a **functional relation**
- $\forall a \in A, \exists! y \in B: (x, y) \in R$
- $(x, y) \in R \leftrightarrow xRy$
- Examples:
 - $A = B = [-1, 1], R = \{(x, y): x^2 + y^2 = 1\}$
 - $A = [-1, 1] \ B = [0, 1], R = \{(x, y): x^2 + y^2 = 1\}$
 - $A = B = \mathbf{R}, R = \{(x, y), y = x^3\}$

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Permutations

A permutation is a bijection from A to A

Composition:

- if f and g are two functions where the values taken on by f are all in the domain of g
 - equivalently, $f: A \rightarrow B$ and $g: C \rightarrow D$ and $f(a) \in C$ for all a in A
- The **composition** of f and g ($gf: A \rightarrow D$, or $g \circ f: A \rightarrow D$) is defined as $(gf)(x) = g(f(x))$.

If f and g are permutations of A , then fg is a permutation of A , and so is f^{-1} .

If f is a permutation of A , we write f^2 instead of ff .

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Cycles

Look at an element x in a permutation f of A

- Look at the sequence of values $x, f(x), f^2(x), \dots$
- Eventually, it will repeat (how long, max?)

If p is the first value for which $fp(x) = x$, then the **cycle** (of length p) containing x is:

- $(x, f(x), f^2(x), f^3(x), \dots, f^{p-1}(x))$

Any permutation can be rewritten as a set of cycles:

- If f maps $(1, 2, 3, 4, 5, 6, 7, 8, 9)$ to $(2, 4, 8, 1, 5, 9, 3, 7, 6)$, then the cycles are:
 - $(1, 2, 4), (3, 8, 7), (5), (6, 9)$
- Can leave off cycles of length 1 (if domain is clear):
 - $(1, 2, 4), (3, 8, 7), (6, 9)$
- Inverse of the function is the cycles in reverse order:
 - $(4, 2, 1), (7, 8, 3), (9, 6)$

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Powers of Permutations

To calculate the power of a function f^i , calculate the power of each cycle

- Calculate final result for each element (using $i \bmod \text{cycle-length}$)
- 1-cycles stay 1-cycles
- For example, if $f = (1, 2, 4), (3, 8, 7, 10), (5), (6, 9)$
 - $f^{10} = (1, 2, 4), (3, 7), (8, 10), (5), (6), (9)$
- If all cycle lengths divide i evenly, f^i is the identity permutation
- If f is a permutation of A , then $f^{|A|!}$ is the identity permutation

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Functions

Given $f: A \rightarrow B$

- **Inverse Image** of f :

$$f^{-1}(b) = \{a: a \text{ in } A \text{ and } f(a) = b\}$$

- **Coimage** of f : partition of A where all elements in a block map to the same element of B
 $\text{Coimage}(f) = \{f^{-1}(b): b \text{ in } \text{Image}(f)\}$

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Partition

Given a set $S = \{a_1, a_2, \dots, a_n\}$:

- A partition P is a set of subsets of S such that:
 - the elements of P **cover** S
 - The elements of P are **pairwise disjoint**
 - No element of P is empty
 - The elements of P are called the **blocks** of the partition

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Number of Partitions

Given $S = \{a_1, a_2, \dots, a_n\}$:

- How many partitions of S are there?
- How many partitions of size k are there?
 - If a_n is in a new block:
 - If a_n is in an existing block
- $S(n, k)$ is the **Stirling number of the second kind**
- $S(n, k) =$

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Calculating $S(n, k)$

	1	2	3	4	5
1					
2					
3					
4					
5					

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Number of functions

Given sets A and B , how many functions are there of the form $f: A \rightarrow B$ where $|\text{Image}(f)| = k$

- How many partitions of A are there of size k ?
- How many images are there of size k ?
- How many ways to map the blocks of $\text{Coimage}(f)$ to $\text{Image}(f)$?

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