

CSE 20—Discrete Math

Winter, 2006

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Sets

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Counting

Given the set $S = \{x, y, z\}$

How many ways to list (without duplicates) all elements of S ?

- Permutations of S

How many ways to list (with duplicates) all elements of S ?

How many ways to construct a k -list of distinct elements from S ?

How many ways to construct a 2-set of (distinct) elements?

How many ways to construct a 2-multiset of elements?

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Permutations and Combinations

Permutation of n objects: ordering of all n objects

r -permutation: ordering of r of n objects

- Order matters
- $P(n, 2) = n(n-1)$
 $P(n, 3) = n(n-1)(n-2)$
 $P(n, r) = n(n-1)(n-2)\dots(n-(r-1))$
 $P(n, n) = n(n-1)\dots(2)(1) = n!$
- $P(n, r) = n(n-1)(n-2)\dots(n-(r-1)) = n!/(n-r)! = n^{\underline{r}} = (n)_r$

r -combination of n objects: choosing r of n objects (unordered)

- $C(n, r) = \# \text{ of } r\text{-permutations} / \# \text{ of ways to order } r \text{ items}$
 $= P(n, r) / P(r, r)$
 $= n!/(n-r)! / r!$
 $= n!/(n-r)!r!$
- $C(n, r) = C(n, n-r)$

$$\binom{n}{r} = C(n, r) = \frac{n(n-1)\dots(n-(r-1))}{r!} = \frac{n!}{r!(n-r)!}$$

Example Problem

How many ways are there to rank n candidates for the job of chief wizard?

If rankings are chosen at random, what is the probability that the fifth candidate is in second place?

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Example Problem

If a 5-card hand is chosen at random, what is the probability of a flush (all 5 cards of the same suit)?

- How many 5-card hands are there?
- How many 5-card hands are flushes?

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Binomial Theorem

Let's look at $(x+y)^n$

$$(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

- How many terms of the form $x^{3-k}y^k$ in $(x+y)^3$?

How many ways to choose k **x**s and $(3-k)$ **y**s from 3 total?

- $C(3, k)$

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Binomial Theorem

Let's look at $(x+y)^n$

- What is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$?

$$\binom{n}{k}$$

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{k}x^{n-k}y^k + \dots + \binom{n}{n}y^n$$

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Binomial Identities

Symmetry Identity

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Fundamental Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Algebraic Proof
- Proof by Pascal's triangle
- Proof by combinatorial argument

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Pascal's Triangle

Number of paths equals $C(n, k)$

	k=0	k=1	k=2	k=3	k=4
n=0					
n=1					
n=2					
n=3					
n=4					

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Functions

Give two sets A and B, a function is a rule that tells how to find a unique $b \in B$ for each $a \in A$.

- It is shown as $f: A \rightarrow B$
- A is the *domain*
- B is the *range* (or *codomain*)
- The set of **all** functions from A to B is written B^A .
 - $f \in B^A \leftrightarrow f: A \rightarrow B$
- Given a set $S \subseteq A$, $f(S) = \{f(x) : x \in S\}$
 - $f(\emptyset) =$
 - $f(A)$ is called *image*

Ways of showing the rule (given domain and range: $A = \{\text{red, green, blue}\}$, $B = \{0..28\}$):

- One-line notation
 - If A is ordered, give the corresponding values of B:
 - Ordering of A = (red, green, blue), $f = (5, 8, 27)$
- Two-line notation
 - Give ordering of A and values of B at the same time:
 - $f = \begin{pmatrix} \text{red} & \text{green} & \text{blue} \\ 5 & 8 & 27 \end{pmatrix}$
- Set notation: $\{(\text{red}, 5), (\text{blue}, 27), (\text{green}, 8)\}$

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Types of Functions

Surjection (onto): Every value in the codomain is taken on at **least once**

- $\forall b \in B, \exists a \in A: f(a) = b$

Injection (one-to-one): Every value in the codomain is taken on at **most once**.

- $\forall a \in A, b \in B, f(a) = f(b) \rightarrow a = b$

Bijection (one-to-one and onto): Every value in the codomain is taken on **exactly once**.

- $\forall b \in B, \exists \text{ unique } a \in A: f(a) = b$

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Example

Let

- S be the set of students attending UCSD
- I be the set of student ID numbers for those students
- D be the set of dates (MM/DD/YYYY) for the past 100 years
- G be the set of possible grade point averages between 2.0 and 3.5 (rounded to the nearest tenth)

Is the following an injection, bijection, or surjection, or none?

- The domain is S, the codomain is I and the function maps each student to his/her ID number
- The domain is S, the codomain is D and the function maps each student to his or her birthday
- The domain is D, the codomain is I and the function maps each date to the ID number of a student born on that date. If more than one, the lexicographically least ID number is chosen.
- The domain is S, the codomain is G and the function maps each student to his or her GPA rounded to the nearest tenth.
- The domain is G, the codomain is I and the function maps each GPA to the ID number of a student with that GPA. If there is more than one, the lexicographically least ID number is chosen.

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Example

Bijjective, injective, surjective, or neither?

- $f: \mathbf{R} \rightarrow \mathbf{R} \ f(x) = x$
- $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+ \ f(x) = x^2$
- $f: \mathbf{R} - \{0\} \rightarrow \mathbf{R}^+ \ f(x) = x^2$
- $f: \mathbf{R} \rightarrow \mathbf{R} \ f(x) = x^2$
- $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+ \ f(x) = \sqrt{x}$
- $f: \mathbf{R} \rightarrow \mathbf{R} \ f(x) = (x-1)x(x+1)$
- $f: \mathbf{R} \rightarrow \mathbf{R} \ f(x) = e^x$
- $f: \mathbf{R} \rightarrow \mathbf{R}^+ \ f(x) = e^x$

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Bijections

Given two sets A and B

- $|A| = |B|$ iff there exists a bijection $f: A \rightarrow B$

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Inverse Functions

If $f: A \rightarrow B$ is a bijection, then f^{-1} is the **inverse** of f . $f^{-1}: B \rightarrow A$

- if $f(a)=b$, then $f^{-1}(b) = a$

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Numbers of Functions

How many functions of the form $f: A \rightarrow B$?

- Equivalently, what is $|B^A|$?
- How many choices for first element of A?
- How many choices for second element of A?
- ...
- How many choices for last element of A?

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