CSE 21 Homework 1 Solutions
Chapter1

Problem 1.26 Find all partitions of $X = \{a, b, c, d\}$.

Solution In the text. Note: if $S(n, k)$ is the number of partitions of n things into k blocks then

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

Summing over all k, $1 \leq k \leq n$ gives $B(n)$. Note here $n = 4$ and $B(4) = 15$. Can you compute this?

Problem 1.49 Let $A = B \cap C$. Determine, which of the statements are true: a. $AXA = (BXB) \cap (CXC)$ b. $AXA = (BXC) \cap (CXB)$

Solution Both are true.

Problem 1.50 Prove: $AX(B \cup C) = (AXB) \cup (AXC)$.

Solution Here to show equality we need to show that

$$AX(B \cup C) \subset (AXB) \cup (AXC)$$

and

$$(AXB) \cup (AXC) \subset AX(B \cup C).$$

I will do one direction and the other is done using the exact same technique. So, assume the ordered pair

$$(x, y) \in AX(B \cup C).$$

Then,

$$x \in A \text{ and } y \in B \text{ and/or } y \in C.$$

Thus,

$$x \in A \text{ and } y \in B \text{ and/or } x \in A \text{ and } y \in C.$$

This is what we wanted to show

$$(x, y) \in (AXB) \cup (AXC).$$

Now,

$$AX(B \cup C) \subset (AXB) \cup (AXC).$$

What if $A = B \cap C \cap D$, then what is $A \times A$?

Problem 1.51 Let $A_n = \{x : x = in\}$ for some $i, n \in \aleph$. Find $A_2 \cap A_7$, $A_6 \cap A_8$, $A_3 \cup A_{12}$, $A_3 \cap A_{12}$, $A_s \cup A_{st}$ where $s, t \in \aleph$, $A_s \cap A_{st}$ where $s, t \in \aleph$, and
Prove: If $J \subset \aleph$ is infinite, then

$$\bigcap_{i \in J} A_i = \emptyset.$$

Solution

1. $A_2 \cap A_7 = A_{14}$.
2. $A_6 \cap A_8 = A_{24}$.
3. $A_3 \cup A_{12} = A_3$.
4. $A_3 \cap A_{12} = A_{12}$.
5. $A_s \cup A_{st} = A_s$ where $s, t \in \aleph$.
6. $A_s \cap A_{st} = A_{st}$ where $s, t \in \aleph$.
7. Prove: If $J \subset \aleph$ is infinite, then

   $$\bigcap_{i \in J} A_i = \emptyset.$$

   To do this we will assume that

   $$\bigcap_{i \in J} A_i \neq \emptyset$$

   and show that this implies that J is finite.
   So, if

   $$\bigcap_{i \in J} A_i \neq \emptyset$$

   then,

   $$\bigcap_{i \in J} A_i = A_\zeta,\ \zeta \in \aleph.$$

   Now,

   $$A_\zeta = A_{1 \times 2 \times \ldots \times J} \rightarrow J = \frac{\zeta}{1 \times 2 \times \ldots \times j - 1}$$

   Thus, J is finite. Here I proved the statement using the contrapositive.

Problem 1.52 Solution is in the text

Problem 1.53 Solution is also in the text.

Problem 1.55 Let $A_1, \ldots, A_m$ and $B_1 \ldots B_n$ be two partitions of a set X. Show that the collection of sets

$$A_i \cap B_j : \ i = 1, \ldots, m, \ j = 1, \ldots, n$$

is also a partition of the set X.

Solution First we will show that if

$$(p, q) \neq (r, s)$$

then

$$(A_p \cap B_q) \cap (A_r \cap A_s) = \emptyset.$$

So, if $(p, q) \neq (r, s)$ then either $p \neq r$ or $q \neq s$ or both. Suppose, without loss of generality that $p \neq r$. Then, by the communitivity of set intersections

$$(A_p \cap B_q) \cap (A_r \cap A_s) = A_p \cap A_r \cap B_q \cap B_s.$$

Now, since $p \neq r$ we get

$$A_p \cap A_r = \emptyset.$$

Thus,

$$(A_p \cap B_q) \cap (A_r \cap A_s) = \emptyset.$$

Now we must show

$$\bigcup_{i,j} A_i \cap B_j = X.$$

Fix i and take the union over all j. Now by the distributive law of intersection over union

$$\bigcup_{i,j} A_i \cap B_j = A_i \cap (\bigcup_j B_j).$$

Since, $B_1, \ldots, B_m$ is a partition

$$\bigcup_j B_j = X.$$

Thus,

$$A_i \cap (\bigcup_j B_j) = A_i \cap X = A_i.$$

So,

$$\bigcup_{i,j} A_i \cap B_j = \bigcup_i A_i = X.$$

Problem 1.56 Prove that for any indexed class $\{A_i : i \in I\}$ and any set B, that
a. $B \cup (\bigcap_i A_i) = \bigcap_i (B \cup A_i)$.

Solution Note I will only do part (a) but (b) is done in the same exact way.

a. So,

$$x \in B \cup (\bigcap_i A_i)$$

if and only if

$$x \in B \text{ or } x \in \bigcap_i A_i$$

if and only if

$$x \in B \text{ or } x \in \bigcap_i A_i \ \forall\ i\ \in\ I$$

if and only if

$$x \in B \cup A_i \ \forall\ i\ \in\ I$$

if and only if

$$x \in \bigcap_i (B \cup A_i).$$

Problem 1.58 Show that each of the following is an algebra of subsets of U: i. $\mathcal{A} = \{\emptyset, U\}$; ii. $\mathcal{B} = \{\emptyset, A, A^c, U\}$; iii. $P(U)$ the power set of U.

Definition A nonempty class $\mathcal{A}$ of subsets of U is an algebra of sets if

1. the complement of any set in $\mathcal{A}$ is also in $\mathcal{A}$ and

2. the union of any finite (countable) number of sets is also in $\mathcal{A}$.

Solution Here just use the definition and for each nonempty class of subsets show (1) and (2) above hold. For example for i. (1) is trivial as is (2). If you still cannot do this problem see 1.59 which I will do in detail.

Problem 1.59 Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of subsets of U. Prove that the intersection $\mathcal{A} \cap \mathcal{B}$ is also an algebra of subsets of U.

Solution Let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ be the intersection of two algebras. Prove $\mathcal{C}$ is an algebra.

1. Suppose, that $x \in \mathcal{C}$ then $x \in \mathcal{A}$ and $x \in \mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are both algebras $X^c \in \mathcal{A}$ and $X^c \in \mathcal{B}$. Thus, $X^c \in \mathcal{C} = \mathcal{A} \cap \mathcal{B}$.

2. Now suppose that $X_1, \ldots, X_n \in \mathcal{C}$. Then for each $i : i = 1, \ldots, n$ $X_i \in \mathcal{A}$ and $X_i \in \mathcal{B}$. Thus, $X_1, \ldots, X_n \in \mathcal{A}$ and $X_1, \ldots, X_n \in \mathcal{B}$. Since, $\mathcal{A}$ and $\mathcal{B}$ are both algebras, the $\bigcap_i X_i$ is in both $\mathcal{A}$ and $\mathcal{B}$. Thus $\bigcap_i X_i$ is in $\mathcal{C}$.