CSE 20 Midterm 2

Note: Please write your full name as it appears on the class roster and include your student id number: A01010101 for example. If you do not include both there is a chance your score will be recorded incorrectly.

Name:

Student I.D.:

Problem 1 Prove or disprove: $\log_2 5$ is irrational.

Solution Suppose it is rational, then

$$log_2 5 = \frac{a}{b}.$$

$$\rightarrow 2^{\frac{a}{b}} = 5.$$

So, if we multiply the exponent by the power b,

$$\rightarrow 2^a = 5^b.$$

Siince this is impossible, i.e. there does not exist $a, b : 2^a = 5^b$, our assumption that the number is rational is false, and thus it must be irrational.

Problem 2 Solution

$$2^{10} = 1024$$

and

$$10^3 = 1000.$$

This is our counterexample.

Problem 3 Simplify the expression using definitions and theorems in Ch 5. Please, justify each and every step you make. Do not use diagrams.

$$\big[B - (A \cap B)\big] \cup (A \cap B)$$

Solution

$$\sqsubset B - (A \cap B) \sqsupset \cup (A \cap B) = \sqsubset B \cap (A \cap B)^c \sqsupset \cup (A \cup B)$$
$$= \sqsubset B \cap (A^c \cup B^c) \sqsupset \cup (A \cup B)$$
$$= \sqsubset (B \cap A^c) \cup (B \cap B^c) \sqsupset \cup (A \cup B)$$
$$= \sqsubset (B \cap A^c) \cup \emptyset \sqsupset \cup (A \cup B)$$

$$= (B \cap A^c) \cup (A \cup B)$$
$$= B \cap (A^c \cup A)$$
$$= B \cap U$$
$$= B.$$

Problem 4 Suppose that a_0, a_1, ... is a sequence defined as follows:

$$a_0 = 0,\ a_1 = 2$$

$$a_n = 4a_{n-1} - 3a_{n-2}$$

for all integers $n \geq 2$. Prove using strong mathematical induction that

$$a_n = -1 + 3^n$$

for all integers $n \geq 0$.

Solution Base case:

$$a_1 = -1 + 3 = 2.$$

Inductive hypothesis: Suppose $a_i = -1 + 3^i$ for all $0 \leq i < k$.

Now show,

$$a_k = -1 + 3^k.$$

So,

$$a_k = 4a_{k-1} - 3a_{k-2}$$

by definition

$$= 4(-1 + 3^{k-1}) - 3(-1 + 3^{k-2})$$

$$= -4 + 4 \times 3^{k-1} + 3 - 3 \times 3^{k-2}$$

$$= -1 + 3^k(4 \times 3^{-1} - 3 \times 3^{-2})$$

$$= -1 + 3^k(\frac{4}{3} - \frac{3}{9})$$

$$= -1 + 3^k.$$

Problem 5 Suppose $A = \{1, 2, ..., n\}$. Let A_n denote the set of all subsets of A that contain n, and A_{n-1} denote the set of all subsets of A that *do not* contain n.

Answer the following questions and give a brief justification or a proof for your answers.

a. Are A_n and A_{n-1} disjoint Yes, if they were not then there exists a set that contains the n^{th} element and doesn't contain it at the same time. Thus the two sets must be disjoint. (i.e. $A_n \cap A_{n-1} = \emptyset$).

b. Compare the sizes of A_n and A_{n-1}.
Well A_{n-1} has n-1 elements and thus

$$|A_{n-1}| = 2^{n-1}.$$

Likewise, for every set that doesnt have the n^{th} element we add it to the set. Thus,

$$|A_{n-1}| = |A_n| = 2^{n-1}.$$

c. How many elements $A_n \cup A_{n-1}$ has and what is its relationship to the set A? Well $A_n \cup A_{n-1} = \mathcal{P}(A)$ which has 2^n elements.