CSE 20: Discrete Mathematics

Spring Quarter 1998

Solutions for Homework Assignment #9

Section 10.3

1. b. The ordered pairs in R are

$$\{ (0,0), (1,1), (1,3), (1,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4), (2,2) \}$$

c. The ordered pairs in R are

$$\{ (0,0), (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4) \}$$

7. The distinct equivalence classes are:

$$\{\emptyset\}, \{\{a\}, \{b\}, \{c\}\}, \{\{a,b\}, \{a,c\}, \{b,c\}\}, \{\{a,b,c\}\}$$

16. There are four distinct classes:

$$\{x \in \mathbf{Z} \mid x = 4k \text{ for some integer } k\}$$
$$\{x \in \mathbf{Z} \mid x = 4k+1 \text{ for some integer } k\}$$
$$\{x \in \mathbf{Z} \mid x = 4k+2 \text{ for some integer } k\}$$
$$\{x \in \mathbf{Z} \mid x = 4k+3 \text{ for some integer } k\}$$

28. *Proof:*

Suppose R is an equivalence relation on a set A, a and b are in A, and $b \in [a]$.
By definition of equivalence class, $b \, R \, a$.
But since R is an equivalence relation, R is symmetric;
hence $a \, R \, b$.

31. *Proof:*

Suppose R is an equivalence relation on a set A, a and b are in A, and $a \in [b]$.
By definition of class, $a \, R \, b$.
We must show that $[a] = [b]$.

To show that $[a] \subseteq [b]$, suppose $x \in [a]$.
[We must show that $x \in [b]$.]
By definition of class, $x\,R\,a$.
By transitivity of R, since $x\,R\,a$ and $a\,R\,b$ then $x\,R\,b$.
Thus by definition of class, $x \in [b]$ *[as was to be shown]*.

To show that $[b] \subseteq [a]$, suppose $x \in [b]$.
[We must show that $x \in [a]$.]
By definition of class, $x\,R\,b$.
But also $a\,R\,b$, and so by symmetry, $b\,R\,a$.
Thus since R is transitive and since $x\,R\,b$ and $b\,R\,a$, then $x\,R\,a$.
Therefore, by definition of class, $x \in [a]$ *[as was to be shown]*.

Since we have prove both subset relations $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we conclude that $[a] = [b]$.

38. a. Haddock's Eyes
    b. The Aged Aged Man
    c. Ways and Means
    d. A-sittin on a Gate

Section 10.4

2. a. 0-equivalence classes:

$$\{s_0, s_1, s_3, s_4,\}, \{s_2, s_5, s_6,\}$$

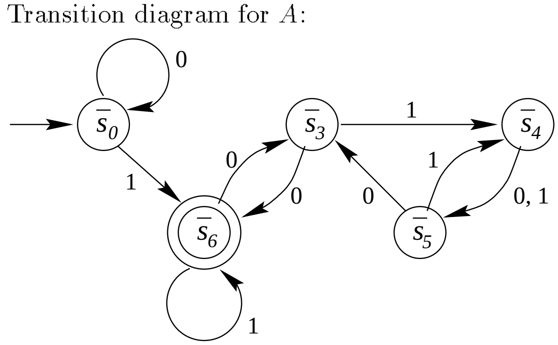
1-equivalence classes:

$$\{s_0, s_1\}, \{s_3\}, \{s_4\}, \{s_2, s_5\}, \{s_6\}$$

2-equivalence classes:

$$\{s_0, s_1\}, \{s_3\}, \{s_4\}, \{s_2, s_5\}, \{s_6\}$$

b. Transition diagram for $\bar{A}$:



8. For A:

   0-equivalence classes:

$$\{s_2, s_4\},\ \{s_0, s_1, s_3\}$$

   1-equivalence classes:

$$\{s_2, s_4\},\ \{s_0, s_1\},\ \{s_3\}$$

   2-equivalence classes:

$$\{s_2, s_4\},\ \{s_0, s_1\},\ \{s_3\}$$

   Therefore, the states of $\bar{A}$ are the 2-equivalence classes of A.

   For A':

   0-equivalence classes:

$$\{s_4'\},\ \{s_0', s_1', s_2', s_3'\}$$

   1-equivalence classes:

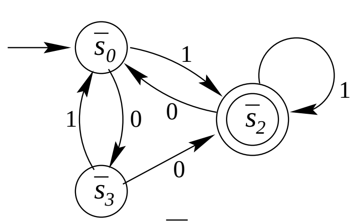
$$\{s_4'\},\ \{s_1', s_3'\},\ \{s_0', s_2'\}$$

   2-equivalence classes:

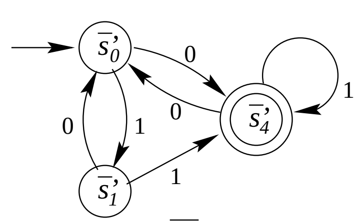
$$\{s_4'\},\ \{s_1', s_3'\},\ \{s_0', s_2'\}$$

   Therefore, the states of $\bar{A}'$ are the 2-equivalence classes of A'.

   According to the text, the automata are equivalent if, and only if, their quotient automata are isomorphic, provided inaccessible states have first been removed. Now A abd A' have no inaccessible states, and $\bar{A}$ has one accepting state and two nonaccepting states as does $\bar{A}'$. But the labels on the arrows connecting the states are different. For instance, in both quotient automata, there is one nonaccepting state which has an arrow going out from it to the accepting state and an arrow going back from the accepting state to it. But for $\bar{A}$, the label on the arrow going to the accepting state is labeled o whereas for $\bar{A}'$ it is labeled 1.



$\overline{A}$



$\overline{A'}$

The nonequivalence of A and A' can also be seen by noting, for example, that the string 00 is accepted by A but not by A'.

14. *Proof:*

Suppose k is an integer such that $k \geq 1$ and states s and t are k-equivalent. Then for all input strings w of length less than or equal to k,

$$N^*(s,w) \text{ is an accepting state} \Leftrightarrow N^*(t,w) \text{ is an accepting state.}$$

Since $k-1 < k$, it follows that for all input strings w of length less than or equal to $k-1$,

$$N^*(s,w) \text{ is an accepting state} \Leftrightarrow N^*(t,w) \text{ is an accepting state.}$$

Hence s and t are $(k-1)$-equivalent.

16. *Proof:*

Suppose s and t are states that are k-equivalent for all integers $k \geq 0$.
Let w be any *[particular but arbitrarily chosen]* input string and let the length of w be l.
Then $l \geq 0$ and so by hypothesis, s and t are R_l-equivalent.
By definition of R_l,

$$N^*(s,w) \text{ is an accepting state} \Leftrightarrow N^*(t,w) \text{ is an accepting state.}$$

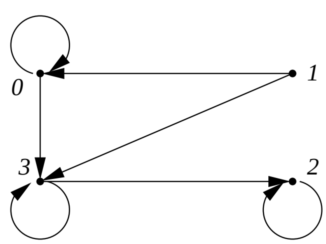
Since the choice of w was arbitrary, we conclude that
for all input strings w,

$$N^*(s,w) \text{ is an accepting state} \Leftrightarrow N^*(t,w) \text{ is an accepting state.}$$

Thus by definition of $*$-equivalence, s and t are $*$-equivalent.

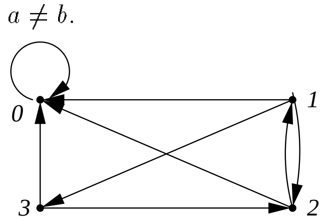
Section 10.5

1. c.



R_3 is antisymmetric: there are no cases where $a\,R_3\,b$ and $b\,R_3\,a$ and $a \neq b$.

d.



R_4 is not antisymmetric: $1\,R_4\,2$ and $2\,R_4\,1$ and $1 \neq 2$.

3. R is not antisymmetric.

   Counterexample:
   Let $s = 0$ and $t = 1$.
   Then $l(s) \leq l(t)$ and $l(t) \leq l(s)$, because both $l(s)$ and $l(t)$ are equal 1, but $s \neq t$.

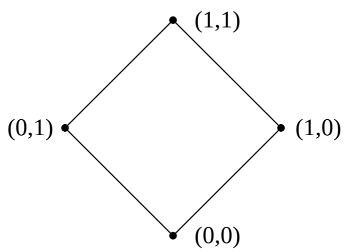
9. R is not a partial order relation because R is not antisymmetric.

   Counterexample:
   Let $m = 2$ and $n = 4$.
   Then $m\,R\,n$ because every prime factor of 2 is a prime factor of 4, and $n\,R\,m$ because every prime factor of 4 is a prime factor of 2.
   But $m \neq n$ because $2 \neq 4$.

10. No.

    Counterexample:
    Let $A = \{1, 2\}$, $R = \{(1, 2)\}$, and $S = \{(2, 1)\}$.
    Then R and S are antisymmetric (by default), but $R \cup S$ is not antisymmetric because $(1, 2) \in R \cup S$ and $(2, 1) \in R \cup S$ and $1 \neq 2$.

19.



25.

greatest element:	$\{0, 1, 2\}$
least element:	$\emptyset$
maximal elements:	$\{0, 1, 2\}$
minimal elements:	$\emptyset$

34. 2, 4, 12, 24 or 3, 6, 12, 24

45. One such total order is

$$\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{d\},\ \{a,b\},\ \{a,c\},\ \{a,d\},\ \{b,c\},\ \{b,d\},\ \{c,d\},$$
$$\{a,b,c\},\ \{a,b,d\},\ \{a,c,d\},\ \{b,c,d\},\ \{a,b,c,d\}.$$

47. a. 33 hours