CSE 20: Discrete Mathematics

Spring Quarter 1998

Solutions for Homework Assignment #8

Section 7.5

1. The solution is given in the textbook.

4. $G \circ F$ is defined by $(G \circ F)(x) = x^{20}$, for all $x \in \mathbb{R}$, because for any real number x,

$$(G \circ F)(x) = G(F(x)) = G(x^5) = (x^5)^4 = x^{20}.$$

   $F \circ G$ is defined by $(F \circ G)(x) = x^{20}$, for all $x \in \mathbb{R}$, because for any real number x,

$$(F \circ G)(x) = F(G(x)) = F(x^4) = (x^4)^5 = x^{20}.$$

   Thus $G \circ F = F \circ G$.

7.

$$\begin{aligned}
(G \circ F)(2) &= G(2^2/3) = G(4/3) = \lfloor 4/3 \rfloor = 1 \\
(G \circ F)(-3) &= G((-3)^2/3) = G(3) = \lfloor 3 \rfloor = 3 \\
(G \circ F)(5) &= G(5^2/3) = G(25/3) = \lfloor 25/3 \rfloor = 8
\end{aligned}$$

9. For each x in $\mathbb{R}^+$,

$$(G \circ G^{-1})(x) = G(G^{-1}(x)) = G(\sqrt{x}) = (\sqrt{x})^2 = x.$$

   Hence $G \circ G^{-1} = i_{\mathbb{R}+}$ by definition of equality of functions.

   For each x in $\mathbb{R}^+$,

$$(G^{-1} \circ G)(x) = G^{-1}(G(x)) = G^{-1}(x^2) = \sqrt{x^2} = x$$

   (because $x > 0$).
   Hence $G^{-1} \circ G = i_{\mathbb{R}+}$ by definition of equality of functions.

12. *Proof:*

   Suppose f is a function from a set X to a set Y.
   For each x in X,

$$(i_Y \circ f)(x) = i_Y(f(x)) = f(x)$$

   by definition of i_Y.
   Hence $i_Y \circ f = f$.

13. *Proof:*

    Suppose $f : X \to Y$ is a one-to-one function with inverse function $f^{-1} : Y \to X$.
    Then for all $y \in Y$,

$$\begin{aligned}(f \circ f^{-1})(y) &= f(f^{-1}(y)) \\ &= f(\text{that element } x \text{ in } X \text{ for which } f(x) \text{ equals y }) \\ &= y \\ &= i_Y(y).\end{aligned}$$

    Hence $f \circ f^{-1} = i_Y$.

15. The solution is given in the textbook.

19. Yes. *Proof:*

    Suppose $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ are functions.
    For each $w \in W$,

$$\begin{aligned}[h \circ (g \circ f)](w) = h((g \circ f)(w)) &= h(g(f(w))) \\ = h \circ g(f(w)) &= [(h \circ g) \circ f](w).\end{aligned}$$

    Hence $h \circ (g \circ f) = (h \circ g) \circ f$ by definition of equality of functions.

20. The solution is given in the textbook.

23. There are two possible proofs for this result:

    Proof 1:
    Suppose $f : X \to Y$ and $g : Y \to Z$ are functions such that $g \circ f = i_X$ and $f \circ g = i_Y$.
    Since both i_X and i_Y are one-to-one and onto, by the results of exercises 15 and 16, both f and g are one-to-one and onto,
    and so by Theorem 7.3.1 and the definition of inverse function, both have inverse functions.
    Now for each $y \in Y$,

$$f^{-1}(y) = \text{that element of } X \text{ such that } f(x) \text{ equals } y.$$

    But

$$f(g(y)) = (f \circ g)(y) = i_Y(y) = y.$$

    Hence $g(y)$ is that element of X such that $f(x)$ equals y,
    and so $g(y) = f^{-1}(y)$.
    It follows by definition of equality of functions that $g = f^{-1}$.

Proof 2:
Suppose $f : X \to Y$ and $g : Y \to Z$ are functions such that $g \circ f = i_X$ and $f \circ g = i_Y$.
Since both i_X and i_Y are one-to-one and onto, by the results of exercises 15 and 16, both f and g are one-to-one and onto,
and so by Theorem 7.3.1 and the definition of inverse function, both have inverse functions.
By Theorem 7.5.2(b),

$$f \circ f^{-1} = i_Y.$$

Since $f \circ g = i_Y$ also,

$$f \circ f^{-1} = f \circ g,$$

and so for all $y \in Y$,

$$(f \circ f^{-1})(y) = (f \circ g)(y).$$

This implies that for all $y \in Y$,

$$f(f^{-1}(y)) = f(g(y)).$$

Since f is one-to-one, it follows that

$$f^{-1}(y) = g(y)$$

for all $y \in Y$,
and so by definition of equality of functions

$$f^{-1} = g.$$

Section 7.6

2. a. Define $f : \mathbb{Z} \to 3\mathbb{Z}$ by the rule $f(n) = 3n$ for all integers n.
The function f is one-to-one because for any integers n_1 and n_2, if $f(n_1) = f(n_2)$, then $3n_1 = 3n_2$ and so $n_1 = n_2$.
Also f is onto because if m is any element in $3\mathbb{Z}$, then $m = 3k$ for some integer k.
But then $f(k) = 3k = m$ by definition of f.
So since there is a function $f : \mathbb{Z} \to 3\mathbb{Z}$ that is one-to-one and onto, $\mathbb{Z}$ has the same cardinality as $3\mathbb{Z}$.

   b. It was shown in Example 7.6.1 that $\mathbb{Z}$ is countably infinite,
which means that $\mathbb{Z}^+$ has the same cardinality as $\mathbb{Z}$.
By part (a), $\mathbb{Z}$ has the same cardinality as $3\mathbb{Z}$.
It follows by the transitive property of cardinality (Theorem 7.6.1(c)) that $\mathbb{Z}^+$ has the same cardinality as $3\mathbb{Z}$.
So $3\mathbb{Z}$ is countably infinite *[by definition of countably infinite],*
and hence $3\mathbb{Z}$ is countable *[by definition of countable].*

3. The solution is given in the textbook.

6. Let B be the set of all bit strings (strings of 0's and 1's).
Define a function $F : \mathbb{Z}^+ \to \mathrm{B}$ as follows:

$$\begin{array}{llll} F(1) = \epsilon, & F(2) = 0, & F(3) = 1, & F(4) = 00, \\ F(5) = 01, & F(6) = 10, & F(7) = 11, & F(8) = 000, \\ F(9) = 001, & F(10) = 010, & & \end{array}$$

and so forth.
At each stage, all the strings of length k are counted before the strings of length $k+1$, and the strings of length k are counted in order of increasing magnitude when interpreted as binary representations of integers.
We can define F more formally as follows:

$$F(n) = \begin{cases} \epsilon & \text{if } n = 1 \\ \text{the } k\text{-bit binary representation of } n - 2^k & \text{if } \lfloor \log_2 n \rfloor = k. \end{cases}$$

For instance, $F(7) = 11$ because $\lfloor \log_2 7 \rfloor = 2$ and the two-bit binary representation of $7 - 2^2 (= 3)$ is 11.

11. *Two examples:*

Define f and g form $\mathbb{Z}$ to $\mathbb{Z}$ as follows: $f(n) = 2n$ and $g(n) = 3n - 2$ for all integers n.
By exercises 8 and 9 of Section 7.3, these functions are one-to-one but not onto.

12. *Two examples:*

Define $F : \mathbb{Z} \to \mathbb{Z}$ by the rule

$$F(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Then F is onto because given any integer m, $m = F(2m)$.
But F is not one-to-one because, for instance, $F(1) = F(3) = 0$.

Define $G : \mathbb{Z} \to \mathbb{Z}$ by the rule $G(n) = \lfloor n/2 \rfloor$ for all integers n.
Then G is onto because given any integer m, $m = \lfloor m \rfloor = \lfloor (2m)/2 \rfloor$.
But G is not one-to-one because, for instance, $G(2) = \lfloor 2/2 \rfloor = 1$ and $G(3) = \lfloor 3/2 \rfloor = 1$ also.

14. a. The solution is given in the textbook.

    b. The fundamental observation is that if one adds up the numbers of ordered pairs along successive diagonals starting from the upper left corner, one obtains a sum of successive integers. The reason is that the number of pairs in the $(m+1)$st diagonal is $m+1$ more than the number in the mth diagonal. We show below hat the value of H for a given pair (m, n) in the diagram is the sum of the numbers of pairs in the diagonals preceding the one containing (m, n) plus the number of the position of (m, n) in its diagonal counting down from the top starting from 0.

Starting in the upper left corner, number the diagonals of the diagram so that the diagonal containing only $(0, 0)$ is 0, the diagonal containing $(1, 0)$ and $(0, 1)$ is 1, the diagonal containing $(2, 0)$, $(1, 1)$ and $(0, 2)$ is 2, and so forth. Within each diagonal, number each ordered pair starting at the top. Thus within diagonal 2, for example, the pair $(2, 0)$ is 0, the pair $(1, 1)$ is 1 and the pair $(0, 2)$ is 2. Each ordered pair of nonnegative integers can be uniquely specified by giving the number of the diagonal that contains it and stating its numerical position within that diagonal. For instance, the pair $(1, 1)$ is in position 1 of diagonal 2, and the pair $(0, 1)$ is in position 0 of diagonal 1. In general, each pair of the form (m, n) lies in diagonal $m+n$, and its position within diagonal $m+n$ is n. Observe that if the arrows in the diagram of exercise 14(a) are followed, the number or ordered pairs that precede $(m+n, 0)$, the top pair of the $(m+n)$th diagonal, is the sum of the numbers of pairs in each of the diagonals from the zeroth through the $(m+n-1)$st. Since there are k pairs in each of the diagonal numbered $k-1$, the number of pairs that precede $(m+n, 0)$ is

$$1+2+3+\ldots+(m+n)=\frac{(m+n)(m+n+1)}{2}$$

by Theorem 4.2.2. Then $\frac{(m+n)(m+n+1)}{2}+n$ is the sum of the number of pairs that precede the top pair of the $(m+n)$th diagonal plus the numerical position of the pair (m, n)within the $(m+n)$th diagonal. Hence

$$H(n)=n+\frac{(m+n)(m+n+1)}{2}$$

is the numerical position of the pair (m, n) in the total ordering of all the pairs if the ordering is begun with 0 at $(0, 0)$ and is continued by following the arrows in the diagram of exercise 14(a).

22. The solution is given in the textbook.

27. *Proof (by contradiction):*

Suppose not.
Suppose S and $\mathcal{P}(S)$ have the same cardinality.
This means that there is a one-to-one, onto function $f: S \to \mathcal{P}(S)$.

Let $A = \{x \in S | x \notin f(x)\}$.
Then $A \in \mathcal{P}(S)$, and since f is onto, there is a $z \in S$ such that $A = f(z)$.
Now either $z \in A$ or $z \notin A$.
In case $z \in A$, then by definition of A, $z \notin f(z) = A$.
Hence in this case $z \in A$ and $z \notin A$ which is impossible.
In case $z \notin A$, then since $A = f(z)$, $z \notin f(z)$ and so z satisfies the condition of membership for the set A which implies that $z \in A$.
nce in this case $z \notin A$ and $z \in A$ which is impossible.
Thus in both cases a contradiction is obtained.
It follows that the supposition is false, and so S and $\mathcal{P}(S)$ do not have the same cardinality.

Section 10.1

5. a. The solution is given in the textbook.

   b. The solution is given in the textbook.

   c. *One possible answer:* 4, 7, 10, -2, -5

   d. *One possible answer:* 5, 8, 11, -1, -4

   e. *Theorem:*

      1. All integers of the form $3k$ are related by T to 0.
      2. All integers of the form $3k+1$ are related by T to 1.
      3. All integers of the form $3k+2$ are related by T to 2.

      Proof of (2):
      Let n be any integer of the form $n = 3k+1$ for some integer k.
      By substitution, $n - 1 = (3k+1) - 1 = 3k$,
      and so by definition of divisibility, $3|(n-1)$.
      Hence by definition of T, $n\,T\,1$.

      The proofs of (1) and (3) are identical to the proof of (2) with 0 and 2 respectively substituted in place of 1.

6. a.

Yes,	$2 \geq 1$.
Yes,	$2 \geq 2$.
No,	$2 \not\geq 3$.
Yes,	$-1 \geq -2$.

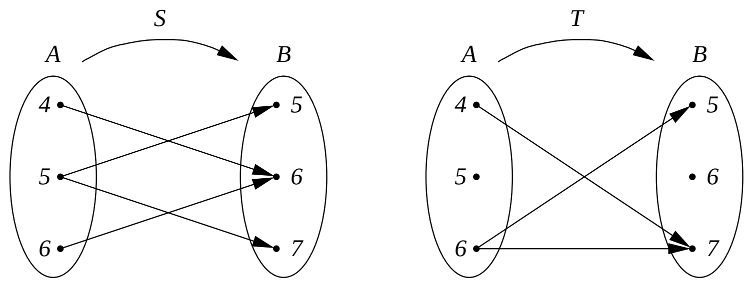
   b. The solution is given in the textbook.

9. a. The solution is given in the textbook.

   b. No, $\{a\}$ has one element and $\{a, b\}$ has two.

   c. Yes, both have one element.

12. a.



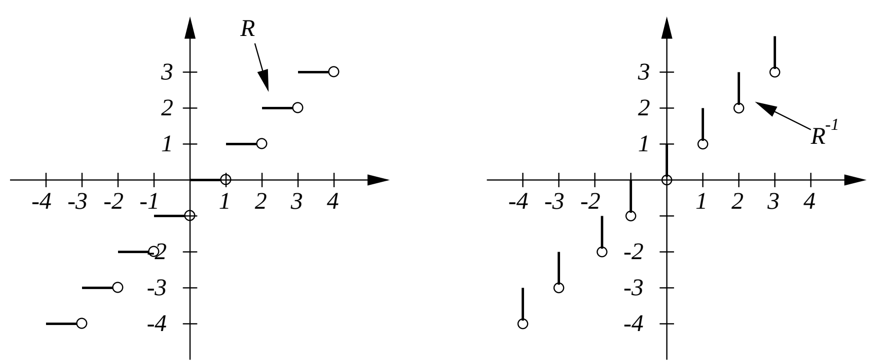
b. S is not a function because $(5,5) \in S$ and $(5,7) \in S$ and $5 \neq 7$.
So S does not satisfy property (2) of the definition of function.
T is not a function both because $(5,x) \notin T$ for any $x \in B$ and because $(6,5) \in T$ and $(6,7) \in T$ and $5 \neq 7$.
So T does not satisfy either property (1) or property (2) of the definition of function.

14. The following sets are all the relations from $\{a,b\}$ to $\{x,y\}$ that are not functions:

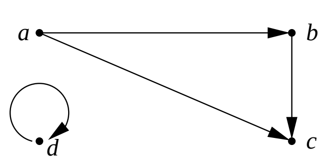
$$\begin{array}{l}\Phi, \quad \{(a,x)\}, \quad \{(a,y)\}, \quad \{(b,x)\}, \quad \{(b,y)\}, \\ \{(a,x),\,(a,y)\}, \quad \{(b,x),\,(b,y)\}, \\ \{(a,x),\,(a,y)\,(b,x)\}, \quad \{(a,x),\,(a,y)\,(b,y)\}, \\ \{(b,x),\,(b,y)\,(a,x)\}, \quad \{(b,x),\,(b,y)\,(a,y)\}, \\ \{(a,x),\,(a,y)\,(b,x)\,(b,y)\}.\end{array}$$

17. The solution is given in the textbook.

20.



24.



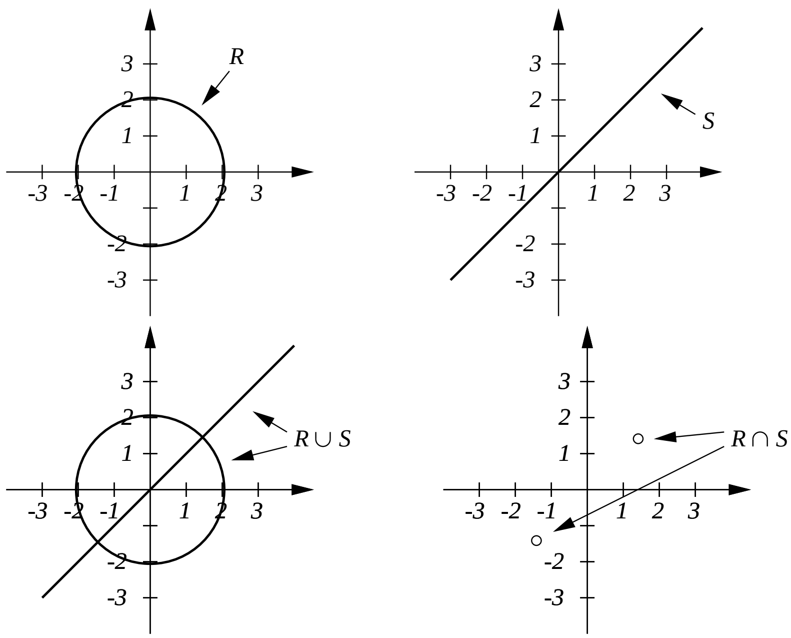
30.

$$\begin{aligned}
A \times B &= \{(-1,1),\ (1,1),\ (2,1),\ (4,1),\ (-1,2),\ (1,2),\ (2,2),\ (4,2)\}\\
R &= \{(-1,1),\ (1,1),\ (2,2)\}\\
S &= \{(-1,1),\ (1,1),\ (2,2),\ (4,2)\}
\end{aligned}$$

$$R \cup S = S$$

$$R \cap S = R$$

32.



To obtain $R \cap S$, solve the system of equations:

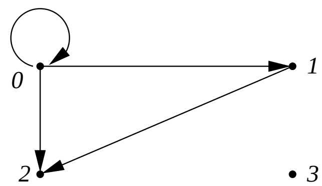
$$\begin{aligned}
x^2 + y^2 &= 4\\
x &= y
\end{aligned}$$

Substituting the second equation into the first gives
$2x^2 = 4 \ \Rightarrow\ x^2 = 2; \Rightarrow\ x = \pm\sqrt{2}$.

Hence $x = y = \sqrt{2}$ or $x = y = -\sqrt{2}$.

Section 10.2

5. a.



b. R_5 is not reflexive because, for example $(1,1) \notin R_5$.

c. R_5 is not symmetric because, for example, $(0,1) \in R_5$ but $(1,0) \notin R_5$.

d. R_5 is transitive.

10.

$$\begin{aligned} S^t = \{&(0,0),\ (0,3),\ (1,0),\ (1,2),\ (2,0),\ (3,2), \\ &(0,2),\ (1,3),\ (2,2),\ (2,3),\ (3,3),\ (3,0)\} \end{aligned}$$

12. The solution is given in the textbook.

17. *O is not reflexive:*
O is reflexive $\Leftrightarrow$ for all integers m, $m\,O\,m$.
By definition of O this means that for all integers m, $m - m$ is odd.
But this is false. As a counterexample, take any integer m.
Then $m - m = 0$, which is even, not odd.

O is symmetric:
Suppose m and n are any integers such that $m\,O\,n$.
By definition of O this means that $m - n$ is odd,
and so by definition of odd $m - n = 2k + 1$ for some integer k.
Now $n - m = -(m - n)$.
Hence by substitution, $n - m = -(2k + 1) = 2(k - 1) + 1$.
It follows that $n - m$ is odd by definition of odd (since $k - 1$ is an integer),
and thus $n\,O\,m$ by definition of O.

O is not transitive:
O is transitive $\Leftrightarrow$ for all integers m, n, and p, if $m\,O\,n$ and $n\,O\,p$ then $m\,O\,p$.
By definition of O this means that for all integers m, n, and p, if $m - n$ is odd and $n - p$ is odd then $m - p$ is odd.
But this is false. As a counterexample, take $m = 1$, $n = 0$, and $p = 1$.
Then $m - n = 1 - 0 = 1$ is odd and $n - p = 0 - 1 = -1$ is also odd, but $m - p = 1 - 1 = 0$ is not odd.
Hence $m\,O\,n$ and $n\,O\,p$ but $m \;\not{O}\; p$.

19. *A is reflexive:*
A is reflexive $\Leftrightarrow$ for all real numbers x, $|x| = |x|$.
But this is true by the reflexive property of equality.

A is symmetric:
[We must show that for any real numbers x and y, if $|x| = |y|$ then $|y| = |x|$.]
But this is true by the symmetric property of equality.

A is transitive:
A is transitive$\Leftrightarrow$ for all real numbers x, y, and z, if $|x| = |y|$ and $|y| = |z|$ then $|x| = |z|$.
But this is true by the transitive property of equality.

23. The solution is given in the textbook.

27. *$\mathcal{R}$ is not reflexive:*
$\mathcal{R}$ is reflexive $\Leftrightarrow$ for all sets $X \in \mathcal{P}(A)$, $X \,\mathcal{R}\, X$.
By definition of $\mathcal{R}$ this means that for all sets $X \in \mathcal{P}(A)$, $X \neq X$.
But this is false for every set in $\mathcal{P}(A)$.
For instance, let $X = \emptyset$. It is not true that $\emptyset \neq \emptyset$.

$\mathcal{R}$ is symmetric:
$\mathcal{R}$ is symmetric $\Leftrightarrow$ for all sets X and Y in $\mathcal{P}(A)$, if $X \,\mathcal{R}\, Y$ then $Y \,\mathcal{R}\, X$.
By definition of $\mathcal{R}$, this means that for all sets X and Y in $\mathcal{P}(a)$, if $X \neq Y$ then $Y \neq X$.
But this is true.

$\mathcal{R}$ is not transitive:
$\mathcal{R}$ is transitive $\Leftrightarrow$ for all sets X, Y, and Z in $\mathcal{P}(A)$, if $X \,\mathcal{R}\, Y$ and $Y \,\mathcal{R}\, Z$ then $X \,\mathcal{R}\, Z$.
By definition of $\mathcal{R}$ this means that for all sets X, Y, and Z in $\mathcal{P}(A)$, if $X \neq Y$ and $Y \neq Z$, then $X \neq Z$.
But this is false as the following counterexample shows.
Since $A \neq \emptyset$, there exists an element x in A.
Let $X = \{x\}$, $Y = \emptyset$, and $Z = \{x\}$.
Then $X \neq Y$ and $Y \neq Z$, but $X = Z$.

32. *$\mathcal{R}$ is reflexive:*
$\mathcal{R}$ is reflexive $\Leftrightarrow$ for all elements $(x, y) \in \mathbb{R} \times \mathbb{R}$, $(x, y) \,\mathcal{R}\, (x, y)$.
By definition of $\mathcal{R}$, this means that for all elements $(x, y) \in \mathbb{R} \times \mathbb{R}$, $y = y$.
But this is true.

$\mathcal{R}$ is symmetric:
[We must show that for all elements (x_1, y_1), $(x_2, y_2) \in \mathbb{R}\times\mathbb{R}$, if $(x_1, y_1) \,\mathcal{R}\, (x_2, y_2)$ then $(x_2, y_2) \,\mathcal{R}\, (x_1, y_1)$.]
Suppose (x_1, y_1), $(x_2, y_2) \in \mathbb{R} \times \mathbb{R}$ such that $(x_1, y_1) \,\mathcal{R}\, (x_2, y_2)$.
By definition of $\mathcal{R}$, this means that $y_1 = y_2$.
By symmetry of equality, $y_2 = y_1$.
So by definition of $\mathcal{R}$, $(x_2, y_2) \,\mathcal{R}\, (x_1, y_1)$.

$\mathcal{R}$ is transitive:
[We must show that for all elements (x_1, y_1),(x_2, y_2), $(x_3, y_3) \in \mathbb{R} \times \mathbb{R}$, if $(x_1, y_1) \,\mathcal{R}\, (x_2, y_2)$ and $(x_2, y_2) \,\mathcal{R}\, (x_3, y_3)$ then $(x_1, y_1) \,\mathcal{R}\, (x_3, y_3)$.]

Suppose (x_1, y_1), (x_2, y_2), $(x_3, y_3) \in \mathbb{R} \times \mathbb{R}$ such that $(x_1, y_1)\,\mathcal{R}\,(x_2, y_2)$ and $(x_2, y_2)\,\mathcal{R}\,(x_3, y_3)$.
By definition of $\mathcal{R}$, this means that $y_1 = y_2$ and $y_2 = y_3$.
By transitivity of equality, $y_1 = y_3$.
Hence by definition of $\mathcal{R}$, $(x_1, y_1)\,\mathcal{R}\,(x_3, y_3)$.

37. a. The solution is given in the textbook.

b. A reflexive relation must contain (a, a) for all eight elements $a \in A$. Any subset of the remaining 56 elements of $A \times A$ (which has a total of 64 elements) can be combined with these eight to produce a reflexive relation. Therefore, there are as many reflexive binary relations as there are subsets of a set of 56 elements, namely 2^{56}.

c. The solution is given in the textbook.

d. Form a relation that is both reflexive and symmetric by a two-step process:

1. Pick all eight elements of the form (x, x) where $x \in A$.
2. Pick a set of pairs of elements of the form (a, b) and (b, a).

There is just one way to perform step 1, and, as explained in the answer to part (c), there are 2^{28} ways to perform step 2.
Therefore, there are 2^{28} binary relations on A that are reflexive and symmetric.

45. R_1 is not irreflexive because $(0, 0) \in R_1$.

R_1 is not asymmetric because $(0, 1) \in R_1$ and $(1, 0) \in R_1$.

R_1 is not intransitive because $(0, 1) \in R_1$ and $(1, 0) \in R_1$ and $(0, 0) \in R_1$.

48. R_4 is irreflexive.

R_4 is not asymmetric because $(1, 2) \in R_4$ and $(2, 1) \in R_4$.

R_1 is intransitive.