# CSE 20: Discrete Mathematics

Spring Quarter 1998

Solutions for Homework Assignment #5

**Section 7.1**

3. c. This one doesn't. The 4 does not have a unique image.

   d. This does. Every element of $X$ has exactly one image in $Y$.

6. This problem is solved in the book.

9. No. For instance, let $F$ and $G$ be defined by the rules

$$F(x) = x \quad \text{and} \quad G(x) = 0$$

Then

$$(F - G)(2) = 2 - 0 = 2 \quad \text{whereas} \quad (G - F)(2) = 0 - 2 = -2$$

and $2 \neq -2$.

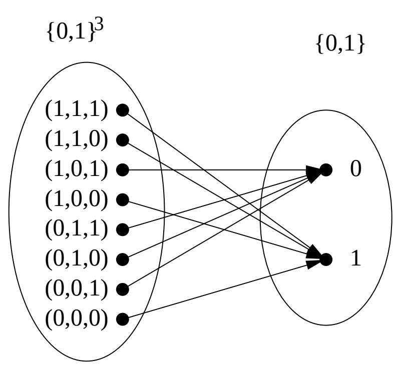
11. b. Define $F : \mathbf{Z}^{nonneg} \to \mathbf{R}$ as follows: for each nonnegative integer $n$, $F(n) = (-1)^n(2n)$.

17. *Proof:* Let $b$ be any positive real number with $b \neq 1$. Then $b^0 = 1$, and so $\log_b 1 = 0$. (Note that we do not allow $b$ to equal 1 here because $\log_b$ is not defined for $b = 1$.)

20. a. This problem is solved in the book.

    b. $p_2(2, y) = y$, $p_2(5, x) = x$. The range of $p_2$ is $\{x, y\}$.

25. b.



27. a. This problem is solved in the book.

b.

| INPUT | | | OUTPUT |
|---|---|---|---|
| $x_1$ | $x_2$ | $x_3$ | $f$ |
| 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 |

28. This problem is solved in the book.

31. a.

$$\begin{aligned}\chi_A(u)\cdot\chi_B(u) &= \begin{cases} 1\cdot 1 & \text{if } u\in A \text{ and } u\in B\\ 1\cdot 0 & \text{if } u\in A \text{ and } u\notin B\\ 0\cdot 1 & \text{if } u\notin A \text{ and } u\in B\\ 0\cdot 0 & \text{if } u\notin A \text{ and } u\notin B\end{cases}\\ &= \begin{cases} 1 & \text{if } u\in A\cap B\\ 0 & \text{if } u\notin A\cap B\end{cases}\\ &= \chi_{A\cap B}(u)\end{aligned}$$

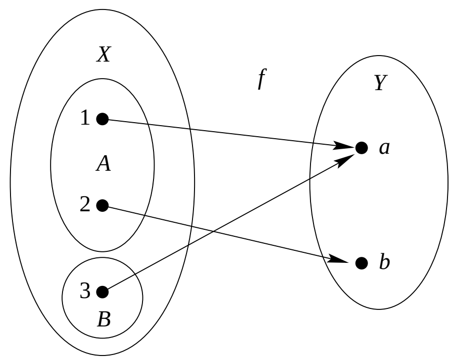
b.

$$\begin{aligned}\chi_A(u)+\chi_B(u)-\chi_A(u)\cdot\chi_B(u) &= \begin{cases} 1+1-1\cdot 1 & \text{if } u\in A \text{ and } u\in B\\ 1+0-1\cdot 0 & \text{if } u\in A \text{ and } u\notin B\\ 0+1-0\cdot 1 & \text{if } u\notin A \text{ and } u\in B\\ 0+0-0\cdot 0 & \text{if } u\notin A \text{ and } u\notin B\end{cases}\\ &= \begin{cases} 1 & \text{if } u\in A \text{ and } u\in B\\ 1 & \text{if } u\in A \text{ and } u\notin B\\ 1 & \text{if } u\notin A \text{ and } u\in B\\ 0 & \text{if } u\notin A \text{ and } u\notin B\end{cases}\\ &= \begin{cases} 1 & \text{if } u\in A\cup B\\ 0 & \text{if } u\notin A\cup B\end{cases}\\ &= \chi_{A\cup B}(u)\end{aligned}$$

40. This property is false.

*Counterexample:* Let $X=\{1,2,3\}$, $Y=\{a,b\}$, $A=\{1,2\}$ and $B=\{3\}$. Then define $f:X\to Y$ by $f(1)=f(3)=a$ and $f(2)=b$.

Illustration:




Now $f(A) = \{a, b\}$, $f(B) = \{a\}$, $f(A - B) = f(\{1, 2\}) = \{a, b\}$, but $f(A) - f(B) = \{a, b\} - \{a\} = \{b\}$. So $f(A) - f(B) \neq f(A - B)$.

**Section 7.1**

6. a. $s_0$, $s_1$, $s_2$, $s_3$
   b. 0, 1
   c. $s_0$
   d. $s_0$
   e.

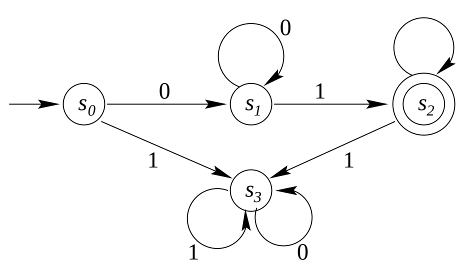
| | | **input** 0 | 1 |
|---|---|---|---|
| $\rightarrow$ $\odot$ | $s_0$ | $s_0$ | $s_1$ |
| | $s_1$ | $s_1$ | $s_2$ |
| **state** | $s_2$ | $s_2$ | $s_3$ |
| | $s_3$ | $s_3$ | $s_0$ |

8. This problem is solved in the book.

10. a. This problem is solved in the book.
    b. $N(s_2, 0) = s_3$, $N(s_1, 0) = s_3$.
    c. This problem is solved in the book.
    d. $N^*(s_2, 11010) = s_3$, $N^*(s_0, 01000) = s_3$

20. This problem is solved in the book.

28.



**Section 7.3**

4. a. $H$ is not one-to-one because $H(b) = y = H(c)$ and $b \neq c$. $H$ is not onto because, for example, $x \in Y$ and $x \neq H(n)$ for any $n \in X$.

   b. $K$ is one-to-one because no two elements of $X$ are sent by $K$ to the same element of $Y$. $K$ is not onto because $z$ is in $Y$ and $z \neq K(x)$ for any $x$ in $X$.

6. a. The solution is given in the textbook.

   b. The solution is given in the textbook.

   c.
$$3 \cdot 2 \cdot 1 = 6.$$
*[There are three choices for where to send the first element of the domain, two choices for where to send the second element (since the function is one-to-one, the second element cannot go to the same place as the first), and one choice for where to send the third, which cannot go the same place as the first two.]*

   d.
$$5 \cdot 4 \cdot 3 = 60.$$
*[There are five choices for where to send the first element of the domain, four choices for where to send the second element, and three choices for where to send the third, which cannot go the same place as the first two.]*

   e. The answer is
$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-m+1)$$
because there are $n$ choices for where to send the first element of the domain, $n-1$ for where to send the second (since it cannot go the same place as the first), $n-2$ for where to send the third (since it cannot go the same place as either of the first two), and so forth. At the time an image is chosen for the $m$th element of the domain, the other $m-1$ elements of the domain have all been sent to $m-1$ distinct elements of the co-domain, and so there are $n-(m-1) = n-m+1$ choices for where to send the $m$th element.

8. The solution is given in the textbook.

10. a. The solution is given in the textbook.

   b. *$K$ is onto:*
Suppose $y$ is any element of $\mathbb{R}^{\text{nonneg}}$. Let $x = \sqrt{y}$. Then $x$ is a real number because $y \geq 0$, and by definition of $K$,
$$K(x) = K\left(\sqrt{y}\right) = \left(\sqrt{y}\right)^2 = y.$$

15. $f$ is one-to-one.

*Proof:*

Let $x_1$ and $x_2$ be any real numbers other than -1, and suppose that

$$f(x_1) = f(x_2).$$

By definition of $f$,

$$\frac{x_1 - 1}{x_1 + 1} = \frac{x_2 - 1}{x_2 + 1}.$$

Cross-multiplying gives

$$(x_1 - 1)(x_2 + 1) = (x_2 - 1)(x_1 + 1),$$

or, equivalently,

$$x_1 x_2 + x_1 - x_2 - 1 = x_1 x_2 + x_2 - x_1 - 1.$$

Adding $1 - x_1 x_2$ to both sides gives

$$x_1 - x_2 = x_2 - x_1,$$

or, equivalently,

$$2x_1 = 2x_2.$$

Dividing both sides by 2 gives

$$x_1 = x_2.$$

19. a. The solution is given in the textbook.

    b. *$F$ is not onto:*
    The number 4 is in $\mathbb{Z}$ but $F(A) \neq 4$ for any set $A$ in $\mathcal{P}(\{a, b, c\})$ because no subset of $\{a, b, c\}$ has four elements.

25. Suppose $b$, $x$ and $y$ are any positive real numbers with $b \neq 1$.
Let $u = \log_b(x)$ and $v = \log_b(y)$.
By definition of logarithm, $x = b^u$ and $y = b^v$.
Then $x \cdot y = b^u \cdot b^v = b^{u+v}$ by property (7.3.1).
Applying the definition of logarithm to the extreme parts of this last equation gives $\log_b(x \cdot y) = u + v = \log_b(x) + \log_b(y)$.

27. The solution is given in the textbook.

28. When $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both onto, it need not be the case that $f + g$ is onto.

*Counterexample:*

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ and $g(x) = -x$ for all $x \in \mathbb{R}$.
Then both $f$ and $g$ are onto, but

$$(f+g)(x) = f(x) + g(x) = x + (-x) = 0 \quad \text{for all } x,$$

and so $f+g$ is not onto.

31. The solution is given in the textbook.

36. By the result of exercise 9(b), $G$ is one-to-one. $G$ is also onto.
For if $y$ is any real number, let $x = (y+2)/3$.
Then $x$ is a real number, and by definition of $G$,

$$G(x) = G(\frac{y+2}{3}) = 3 \cdot \frac{y+2}{3} - 2 = (y+2) - 2 = y.$$

Hence $G$ is onto.
Since $G$ is both one-to-one and onto, it is a one-to-one correspondence, and the above calculation shows that for all $y \in \mathbb{R}$,

$$G^{-1}(y) = \frac{y+2}{3}.$$

**Section 7.4**

2. a. No. For example, thirteen hearts could be selected: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A. No two of these are of the same denomination.

   b. Yes. Let $X$ be the set consisting of the 20 selected cards, and let $Y$ be the 13 possible denominations of cards. Define a function $D$ from $X$ (the pigeons) to $Y$ (the pigeonholes) by the rule $D(x) =$ the denomination of $x$ for all $x$ in $X$. Now $X$ has 20 elements and $Y$ has 13 and $20 > 13$. So by the pigeonhole principle, $D$ is not one-to-one. Hence $D(x_1) = D(x_2)$ for some cards $x_1$ and $x_2$ with $x_1 \neq x_2$. Then $x_1$ and $x_2$ are two distinct cards out of the 20 selected cards that have the same denomination.

6. a. Yes. Let $X$ be the set of seven integers and $Y$ the set of all possible remainders obtained through division by 6, and consider the function $R$ from $X$ (the pigeons) to $Y$ (the pigeonholes) defined by the rule: $R(n) = n = n \ mod \ 6$ (= the remainder obtained by the integer division of $n$ by 6). Now $X$ has 7 elements and $Y$ has 6 elements (0, 1, 2, 3, 4, and 5). Hence by the pigeonhole principle, $R$ is not one-to-one: $R(n_1) = R(n_2)$ for some integers $n_1$ and $n_2$ with $n_1 \neq n_2$. But this means that $n_1$ and $n_2$ have the same remainder when divided by 6.

   b. No. Consider the set $\{1, 2, 3, 4, 5, 6, 7\}$. This set has seven elements no two of which have the same remainder when divided by 8.

8. No. For instance, the five integers 1,2,3,4,5 could be chosen. The sum of any two of these is less than 10 and so no two have a sum of 10.

10. The solution is given in the textbook.

15. There are $n+1$ even integers from 0 to $2n$ inclusive:

$$0(=2\cdot 0),\ 2(=2\cdot 1),\ 4(=2\cdot 2),\ \ldots,\ 2n(=2\cdot n).$$

So a maximum of $n+1$ integers can be chosen, and thus if at least $n+2$ integers are chosen, one is sure to be odd.
Similarly, since there are $n$ odd integers from 0 to $2n$ inclusive,

$$1(=2\cdot 1-1),\ 3(=2\cdot 2-1),\ \ldots,\ 2n-1(=2\cdot n-1),$$

if at least $n+1$ integers are chosen, one is sure to be even.

24. *Proof (by contradiction):*

Suppose there is a finite-state automaton $A$ that accepts $L$. Consider all strings of the form $a^i$ for some integer $i \geq 0$. Since the set of all such strings is infinite and the number of states of $A$ is finite, by the pigeonhole principle at least two of these strings, say $a^p$ and $a^q$ with $p < q$, must send $A$ to the same state, say $s$, when input to $A$ starting in its initial state. Because $A$ accepts $L$, $A$ accepts $a^p b^p$ but not $a^q b^p$. But since $a^p b^p$ is accepted by $A$, inputting $b^p$ to $A$ when it is in state $s$ (after input of $a^p$) sends $A$ to an accepting state. Because $A$ also goes to state $s$ after input of $a^q$, inputting $b^p$ to $A$ after inputting $a^q$ also sends $A$ to an accepting state. Thus $a^q b^p$ is accepted by $A$ and yet it is not accepted by $A$, which is a contradiction. Hence the supposition is false: there is no finite-state automaton that accepts $L$.

27. The solution is given in the textbook.

32. There are two possible ways of proving this:

*Proof (by contradiction):*

Suppose that two or fewer secretaries are each assigned to three or more executives. Then the remaining secretaries are each assigned to two or fewer executives. Since the maximum number of executives to which any secretary can be assigned is four, the maximum number of executives that can be served by the secretaries is obtained when 2 secretaries (the maximum possible) are each assigned to 4 executives (the maximum possible) and the remaining 3 secretaries are each assigned to 2 executives (the maximum possible for that group). The maximum number of executives that can be served by the secretaries is, therefore, $2\cdot 4+3\cdot 2=14$. Thus the five secretaries are assigned to at most 14 executives, and so at least one executive does not have a secretary, which contradicts the fact that each executive is assigned a secretary. Consequently, the supposition that two or fewer secretaries are assigned to three of more executives is false *[and so at least three secretaries are assigned to three or more executives].*

*Proof (direct):*

Let $k$ be the number of secretaries assigned to three or more executives. Because no secretary is assigned to more than four executives, these secretaries are assigned to at most $4k$ executives. Each ot the remaining $5 - k$ secretaries is assigned to at most two executives, and so together they are assigned to at most $2(5 - k) = 10 - 2k$ executives. Therefore the number of executives assigned secretaries is at most $4k + (10 - 2k) = 10 + 2k$. Since 15 executives are assigned secretaries, $15 \leq 10 + 2k$, or $k \geq 5/2$. So, since $k$ is an integer, $k \geq 3$. Hence at least three secretaries are assigned to three or more executives.