CSE 20: Discrete Mathematics

Spring Quarter 1998

Solutions for Homework Assignment #3

Section 4.4

2. *Proof:* Let $P(n)$ be the property "b_n is divisible by 3." We prove by strong mathematical induction that this property is true for all integers $n \geq 1$.

   The property is true for $n = 1$ and $n = 2$: $b_1 = 3$ and $b_2 = 6$ and both 3 and 6 are divisible by 3.

   If the property is true for all integers i with $1 \leq i < k$ then it is true for k:

   Let $k > 2$ be an integer and suppose b_i is divisible by 3 for all integers i with $1 \leq i < k$. *[This is the inductive hypothesis.]* We must show that b_k is divisible by 3.

   By definition of b_k, $b_k = b_{k-2} + b_{k-1}$. Since $k > 2$, $1 \leq k - 2 < k$ and $1 \leq k - 1 < k$, and so by inductive hypothesis, both b_{k-1} and b_{k-2} are divisible by 3.

   Thus *[by exercise 14 of Section 3.3]* the sum of b_{k-1} and b_{k-2}, which is b_k, is also divisible by 3 *[as was to be shown]*.

5. *Proof:* Let $P(n)$ be the inequality "$e_n \leq 3^n$." We prove by strong mathematical induction that this inequality is true for all integers $n \geq 0$.

   The inequality is true for $n = 0$, 1 and 2: $e_0 = 1$, $e_1 = 2$ and $e_2 = 3$ and $1 \leq 3^0$, $2 \leq 3^1$ and $3 \leq 3^3$.

   If the property is true for all integers i with $1 \leq i < k$ then it is true for k:

   Let $k > 2$ be an integer and suppose $e_i \leq 3^i$ for all integers i with $1 \leq i < k$. *[This is the inductive hypothesis.]* We must show that $e_k \leq 3^k$.

   By definition of e_k, $e_k = e_{k-1} + e_{k-2} + e_{k-3}$. Since $k > 2$, $1 \leq k-3 < k$ and so by inductive hypothesis, $e_{k-1} \leq 3^{k-1}$, $e_{k-2} \leq 3^{k-2}$, and $e_{k-3} \leq 3^{k-3}$.

   Adding the inequalities and using the basic laws of algebra gives

$$\begin{aligned} e_{k-1} + e_{k-2} + e_{k-3} &\leq 3^{k-1} + 3^{k-2} + 3^{k-3} \\ &\leq 3^{k-1} + 3^{k-1} + 3^{k-1} \\ &= 3 \cdot 3^{k-1} \\ &= 3^k \end{aligned}$$

   But $e_{k-1} + e_{k-2} + e_{k-3} = e_k$ and so $e_k \leq 3^k$ *[as was to be shown]*.

10. The inductive step fails when $k = 1$. The reason is that to go from $k = 1$ to $k = 2$, one evaluates

$$\frac{r^{k-1} \cdot r^{k-1}}{r^{k-2}}$$

for $k = 1$. But the inductive hypothesis only says that $r^i = 1$ for all i with $0 \leq i < k$ and when $k = 1$ then $k - 2 = -1 < 0$. Therefore it cannot be deduced from the inductive hypothesis that $r^{k-2} = 1$.

11. This problem is solved in the book.

13. The answer is "no".

    Counterexample 1:

    Let $P(n)$ be "$n+24$ is composite." Then $P(0)$, $P(1)$, and $P(2)$ are all true (because 24, 25, and 26 are all composite), and for all integers $k \geq 0$, if $P(k)$ is true, then $P(3k)$ is true (because $3k+24 = 3 \cdot (k+8)$ is composite for every integer k).

    But it is false that $n + 24$ for is composite for all integers $n \geq 0$, e.g. 5+24=29 is prime (note that $5 \neq 3k$ for any integer k).

    Counterexample 2:

    Let $P(n)$ be "$n = 0$, $n = 1$, $n = 2$, or $n = 3r$ for some integer r." Then $P(0)$, $P(1)$, and $P(2)$ are all true (because an *or* statement is true if any component is true), and for all integers $k \geq 3$, if $P(k)$ is true, then $P(3k)$ is true (also by definition of the truth value of an *or* statement and by definition of $P(n)$).

    But $P(n)$ is not true for all integers $n \geq 0$ (because, for instance, $P(4)$ is not true.

    (There may be other counterexamples besides these two.)

Section 4.5

4. *Proof:* Suppose the condition $2^n < (n+2)!$ is true before entry to the loop. Then $2^{n_{old}} < (n_{old} + 2)!$. After execution of the loop, $n_{new} = n_{old} + 1$. So

$$\begin{aligned} 2^{n_{new}} &= 2^{n_{old}+1} \\ &= 2 \cdot 2^{n_{old}} \\ &< 2 \cdot (n_{old} + 2)! \\ &< (n_{old} + 3) \cdot (n_{old} + 2)! \quad \textit{[because } n_{old} \geq 0\textit{]} \\ &= (n_{old} + 3)! \\ &= (n_{new} + 2)! \end{aligned}$$

   Hence $2^{n_{new}} < (n_{new} + 2)!$.

6. This problem is solved in the book.

9. I **Basis Property:** $I(0)$ is the statement "both A and a are even integers or both are odd integers and $a \geq -1$." According to the pre-condition, this statement is true.

II **Inductive Property:** Suppose k is a nonnegative integer such that $G \wedge I(k)$ is true before an iteration of the loop. Hence as execution reaches the top of the loop, $a_{old} > 0$ and $I(k)$ holds (i.e. a_{old} and A are both even or both odd integers, and $a_{old} \geq -1$).

Execution of statement 1 sets a_{new} equal to $a_{old} - 2$. Hence a_{new} has the same parity as a_{old} which is the same as A.

Also, since $a_{old} > 0$, $a_{new} = a_{old} - 2 > 0 - 2 = -2$. But a_{new} is an integer. So since $a_{new} > -2$, $a_{new} \geq -1$.

Hence after the loop iteration, $I(k+1)$ is true.

III **Eventual falsity of guard:** The guard G is the condition $a > 0$. After each iteration of the loop, $a_{new} = a_{old} - 2 < a_{old}$, and so successive iterations of the loop give a strictly decreasing sequence of integer values of a which eventually becomes less than or equal to zero, at which point G becomes false.

IV **Correctness of the post-condition:** Suppose that N is the least number of iterations after which G is false and $I(N)$ is true.

Then (since G is false) $a \leq 0$ and (since $I(N)$ is true) both a and A are even integers or both are odd integers, and $a \geq -1$.

Putting the inequalities together gives $-1 \leq a \leq 0$, and since a is an integer, $a = -1$ or $a = 0$. Since A and a have the same parity, $a = 0$ if A is even and $a = -1$ if A is odd. This is the post-condition.

10. I **Basis Property:** $I(0)$ is the statement

(1) a and b are nonnegative integers with $\gcd(a, b) = \gcd(A, B)$,

(2) at most one of a and b equals zero,

(3) $0 \leq a + b \leq A + B - 0$

According to the pre-condition, A and B are positive integers, and $a = A$ and $b = B$, so $I(0)$ is true.

II **Inductive Property:** Suppose k is a nonnegative integer such that $G \wedge I(k)$ is true before an iteration of the loop. Then as execution reaches the top of the loop, $a_{old} \neq 0$ and $b_{old} \neq 0$ and

(1) a_{old} and b_{old} are nonnegative integers with $\gcd(a_{old}, b_{old}) = \gcd(A, B)$,

(2) at most one of a_{old} and b_{old} equals zero,

(3) $0 \leq a_{old} + b_{old} \leq A + B - k$

After execution of the statement

"**if** $a \geq b$ **then** $a := a - b$ **else** $b := b - a$"

we have the following:

(1) *Case 1* $(a_{old} \geq b_{old})$*:* In this case $a_{new} = a_{old} - b_{old} \geq 0$ and $b_{new} = b_{old} \geq 0$, and so both a_{new} and b_{new} are nonnegative. Also

$$\begin{aligned}\gcd(a_{new}, b_{new}) &= \gcd(a_{old} - b_{old}, b_{old}) \\ &= \gcd(a_{old}, b_{old})\end{aligned}$$

by Lemma 3.8.3. But $\gcd(a_{old}, b_{old}) = \gcd(A, B)$ by (1) above. Hence $\gcd(a_{new}, b_{new}) = \gcd(A, B)$.

Case 2 $(a_{old} < b_{old})$*:* In this case $b_{new} = b_{old} - a_{old} \geq 0$ and $a_{new} = a_{old} \geq 0$, and so both a_{new} and b_{new} are nonnegative. Also

$$\begin{aligned}\gcd(a_{new}, b_{new}) &= \gcd(a_{old}, b_{old} - a_{old}) \\ &= \gcd(a_{old}, b_{old})\end{aligned}$$

by Lemma 3.8.3. But $\gcd(a_{old}, b_{old}) = \gcd(A, B)$ by (1) above. Hence $\gcd(a_{new}, b_{new}) = \gcd(A, B)$.

(2) Because G is true $a_{old} \neq 0$ and $b_{old} \neq 0$, and by (1) above both a_{old} and b_{old} are nonnegative. Hence $a_{old} \geq 0$ and $b_{old} \geq 0$. Since either $a_{new} = a_{old}$ or $b_{new} = b_{old}$, at most one of a_{new} and b_{new} equals zero.

(3) Observe that

$$\begin{aligned}a_{new} + b_{new} &= \begin{cases} a_{old} - b_{old} + b_{old} & \text{if} \quad a_{old} \geq b_{old} \\ a_{old} + b_{old} - a_{old} & \text{if} \quad b_{old} > a_{old} \end{cases} \\ &= \begin{cases} a_{old} & \text{if} \quad a_{old} \geq b_{old} \\ b_{old} & \text{if} \quad b_{old} > a_{old}. \end{cases}\end{aligned}$$

But since $a_{old} \neq 0$ and $b_{old} \neq 0$ and a_{old} and b_{old} are nonnegative integers, $a_{old} \geq 1$ and $b_{old} \geq 1$.
Hence $a_{old} - 1 \geq 0$ and $b_{old} - 1 \geq 0$, and so $a_{old} \leq a_{old} + b_{old} - 1$. But by (3) above $a_{old} + b_{old} \leq A + B - k$. It follows that

$$a_{new} + b_{new} \leq a_{old} + b_{old} - 1 \leq A + B - (k+1).$$

III **Eventual Falsity of Guard:** The Guard G is the condition $a \neq 0$ and $b \neq 0$.

Suppose G is never false.. *[We will derive a contradiction.]* Then the loop would iterate indefinitely. But by II. above, for all integers n, after n iterations of the loop, $a \geq 0$, $b \geq 0$, at most one of a and b equals zero, and $0 \leq a + b \leq A + B - n$.

Hence after $A + B$ iterations,

$$0 \leq a + b \leq A + B - (A + B) = 0.$$

Since $a \geq 0$ and $b \geq 0$, it is $a = b = 0$. But this contradicts the fact that at most one of a and b equals zero. Hence the supposition is false, and so after some iteration G is false.

IV **Correctness of the Post-Condition:** Suppose that N is the least number of iterations after which G is false and $I(n)$ is true.

Then (since G is false) $a = 0$ or $b = 0$, and (since $I(n)$ is true) a and b are nonnegative integers with $\gcd(a, b) = \gcd(A, B)$ and at most one of a and b equals zero. Thus

$$\gcd(a,b) = \begin{cases} \gcd(0,b) & \text{if} \quad a = 0 \\ \gcd(a,0) & \text{if} \quad b = 0 \end{cases}$$

By Lemma 3.8.1. $\gcd(0, b) = b$ and $\gcd(a, 0) = a$. Hence one of a or b is zero and the other is nonzero, and $\gcd(A, B)$ equals whichever of a or b is nonzero.

Section 5.1

5. a. The solution is given in the book.
   b. Yes. Every element in C is contained in A.
   c. Yes. Every element in C is contained in C.
   d. The solution is given in the book.

8. The solution is given in the book.

12. a. The solution is given in the book.
    b. The solution is given in the book.
    c. No. $8 \in D$ because $8 = 3 \cdot 3 - 1$. But $8 \notin A$.
    For if 8 were in A, then $8 = 2i - 1$ for some integer i.
    Solving for i would give $2i = 9$, or $i = 9/2$, which is not an integer.
    Hence $8 \notin A$.
    d. Yes.
    [Show $B \subseteq D$ and $D \subseteq B$:]

    Suppose $n \in B$.
    By definition of B, $n = 3j + 2$ for some integer j.
    But then $n = 3j + 2 = 3j + 3 - 1 = 3(j + 1) - 1$.
    Let $s = j + 1$.
    Then s is an integer and $n = 3s - 1$.
    So by definition of D, $n \in D$.
    Hence any element of B is in D, or, symbolically, $B \subseteq D$.

Conversely, suppose $q \in D$.
By definition of D, $q = 3s - 1$ for some integer s.
But then $q = 3s - 1 = 3s - 3 + 2 = 3(s-1) + 2$ for some integer s.
Let $j = s - 1$.
Then j is an integer and $q = 3j + 2$.
So by definition of B, $q \in B$.
Hence any element of D is in B, or, symbolically, $D \subseteq B$.

Since $B \subseteq D$ and $D \subseteq B$, by definition of set equality $B = D$.

15.

18. c.

$$\begin{aligned} A \times B \times C = \{&(1,u,m),(1,u,n),(1,v,m),(1,v,n),\\ &(2,u,m),(2,u,n),(2,v,m),(2,v,n),\\ &(3,u,m),(3,u,n),(3,v,m),(3,v,n)\} \end{aligned}$$

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$$L = \{11*, 11/, 12*, 12/, 21*, 21/, 22*, 22/\}$$

22.

i	1			2				3					4
j	1	2	3	4	1	2	3	1	2	3	4	5	
$found$	no	yes		no		yes		no					
$answer$	$A \subseteq B$											$A \not\subseteq B$	

Section 5.2

4. a. $A \cup B \subseteq B$
   b. $A \cup B$
   c. $x \in B$
   d. A
   e. or
   f. B
   g. A
   h. B
   i. B

9. *Proof:*

   Suppose A and B are arbitrarily chosen sets.

   Show $(A - B) \cup (A \cap B) \subseteq A$:
   Suppose $x \in (A - B) \cup (A \cap B)$.
   By definition of union, $x \in (A - B)$ or $x \in A \cap B$.

   Case 1 ($x \in A - B$):
   Then by definition of set difference $x \in A$ and $x \notin B$.
   In particular, $x \in A$.

   Case 2 ($x \in A \cap B$):
   Then by definition of intersection $x \in A$ and $x \in B$.
   In particular, $x \in A$.

   Hence in either case, $x \in A$, and so by definition of subset,
   $(A - B) \cup (A \cap B) \subseteq A$.

   Show $A \subseteq (A - B) \cup (A \cap B)$:
   Suppose $x \in A$.
   Either $x \in B$ or $x \notin B$.

   Case 1 ($x \in B$):
   Then since $x \in A$ also, by definition of intersection $x \in (A \cap B)$,
   and so by the inclusion in union property, $x \in (A - B) \cup (A \cap B)$.

   Case 2 ($x \notin B$):
   Then since $x \in A$, by definition of set difference $x \in A - B$,
   and so by the inclusion in union propterty, $x \in (A - B) \cup (A \cap B)$.

   Hence in either case, $x \in (A - B) \cup (A \cap B)$,
   and so by definition of subset, $A \subseteq (A - B) \cup (A \cap B)$.

   Since both set containments have been proved,
   $(A - B) \cup (A \cap B) = A$ by definition of set equality.

11. True.

    Proof:

    Let A, B and C be any sets.
    [Show $(A-B)\cap(C-B) \subseteq (A\cap C)-B$ and $(A\cap C)-B \subseteq (A-B)\cap(C-B)$:]

    Show $(A - B) \cap (C - B) \subseteq (A \cap C) - B$:
    Suppose $x \in (A - B) \cap (C - B)$.
    By definition of intersection, $x \in A - B$ and $x \in C - B$,
    and so by defintion of set difference, $x \in A$ and $x \notin B$ and $x \in C$ and $x \notin B$.
    Thus $x \in A$ and $x \in C$ and $x \notin B$.
    So by definition of intersection, $x \in A \cap C$ and $x \notin B$,
    and by definition of set difference $x \in (A \cap C) - B$.

[Thus $(A - B) \cap (C - B) \subseteq (A \cap C) - B$ by definition of subset.]

Show $(A \cap C) - B \subseteq (A - B) \cap (C - B)$:
Suppose $x \in (A \cap C) - B$.
By definition of set difference, $x \in A \cap C$ and $x \notin B$,
and by definition of intersection, $x \in A$ and $x \in C$ and $x \notin B$.
Thus it is true that $x \in A$ and $x \notin B$ and $x \in C$ and $x \notin B$,
and so by definiton of set difference, $x \in A - B$ and $x \in C - B$,
and by definition of intersection, $x \in (A - B) \cap (C - B)$.
[Thus $(A \cap C) - B \subseteq (A - B) \cap (C - B)$ by definition of subset.]

[Since both subset containments have been proved, $(A - B) \cap (C - B) = (A \cap C) - B$ by definition of set equality.]

15. The solution is given in the book.

20. False. *Counterexample:*

    Let $A = \{1\}$, $B = \{2\}$ and $C = \{1, 3\}$.
    Then $A \nsubseteq B$ and $B \nsubseteq C$ but $A \subseteq C$.

24. True.

    Proof:
    Suppose A, B and C are sets.

    Show $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$:
    Suppose $(x, y) \in A \times (B \cap C)$.
    By definition of Cartesian product, $x \in A$ and $y \in B \cap C$.
    By definition of intersection, $y \in B$ and $y \in C$.
    It follows that both statements "$x \in A$ and $y \in B$" and "$x \in A$ and $y \in C$" are true.
    Hence by definition of Cartesian product, $(x, y) \in A \times B$ and $(x, y) \in A \times C$,
    and so by definition of intersection, $(x, y) \in (A \times B) \cap (A \times C)$.
    [Thus $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$] by definition of subset.

    Show $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$:
    Suppose $(x, y) \in (A \times B) \cap (A \times C)$.
    By definition of intersection, $(x, y) \in A \times B$ and $(x, y) \in A \times C$,
    and so by definition of Cartesian product $x \in A$ and $y \in B$ and also $x \in A$ and $y \in C$.
    Consequently, the statement "$x \in A$ and both $y \in B$ and $y \in C$" is true.
    It follows by definition of intersection that $x \in A$ and $y \in B \cap C$,
    and so by definition of Cartesian product, $(x, y) \in A \times (B \times C)$.
    [Thus $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ by definition of subset.]

[Since both subset containments have been proved, $(A \times B) \cap (A \times C) = A \times (B \cap C)$ *by definition of set equality.]*

26. The solution is given in the book.

29. The solution is given in the book.

36. *Proof:*

    Let sets A and B be given. Then

$$\begin{aligned}
B^c \cup (B^c - A)^c &= (B^c \cup (B^c \cap A^c))^c && \text{by the alternate representation} \\
& && \text{for set difference laws} \\
&= (B^c)^c \cap (B^c \cap A^c)^c && \text{by De Morgan's law} \\
&= B \cap (B^c \cap A^c)^c && \text{by the double complement law} \\
&= B \cap ((B^c)^c \cup (A^c)^c) && \text{by De Morgan's law} \\
&= B \cap (B \cup A) && \text{by the double complement law} \\
&= B && \text{by the absorption law.}
\end{aligned}$$

Section 5.4

4. Since there are no real numbers with negative squares, this sentence is vacuously true, and hence it is a statement.

5. The solution is given in the textbook.

7. Suppose Nixon says (ii) and the only utterance Jones makes about Watergate is (i). Suppose also that apart from (ii) all of Nixon's other assertions about Watergate are evenly split between true and false.

   Case 1 (Statement (i) is true):
   In this case, more than half of Nixon's assertions about Watergate are false, and so (since all of Nixon's other assertions about Watergate are evenly split between true and false) statement (ii) must be false (because it is an assertion about Watergate). So at least one of Jones' statements about Watergate is false. But the only statement Jones makes about Watergate is (i). So statement (i) is false.

   Case 2 (statement (i) is false):
   In this case, half or more of Nixon's assertions about Watergate are true, and so (since all of Nixon's other assertions about Watergate are evenly split between true and false) statement (ii) must be true. But statement (ii) asserts that everything Jones says about Watergate is true. And so in particular, statement (i) is true.

   The above arguments show that under the given circumstances, statements (i) and (ii) are contradictory.

9. No. Suppose there were such a book. If such a book did not refer to itself, then it would belong to the set of all books that do not refer to themselves. But it is supposed to refer to all books in this set, and so it would refer to itself. On the other hand, if such a book referred to itself, then it would belong to the set of books to which it refers and this set only contains books that do not refer to themselves. Thus it would not refer to itself. It follows that the assumption that such a book exists leads to a contradiction, and so there is no such book.