CSE 20: Discrete Mathematics

Spring Quarter 1998

Solutions for Homework Assignment #4

Section 3.6

1. The solution is in the book.

4. *statement:* There is no least positive rational number.

   negation: There is a least positive rational number.

   Proof (by contradiction): Suppose not.
   That is, suppose there is a least positive rational number.
   Call this number r.
   Then r is a real number such that $r > 0$, r is rational, and for all positive rational numbers x, $x \geq r$.

   Let $s = r/2$.*[We will show that s is a positive rational number with $s < r$.]*
   [Show that $0 < s < r$:] Note that if we divide both sides of the inequality $0 < r$ by 2, we obtain $0 < r/2 = s$,
   and if we add r to the inequality $0 < r$ and then divide by 2, we obtain $\frac{0+r}{2} < \frac{r+r}{2}$, or, equivalently, $s = r/2 < r$.
   Hence $0 < s < r$.
   [Show that s is rational:] Note also that since r is rational, $r = a/b$ for some integers a and b with $b \neq 0$ and so $s = \frac{r}{2} = \frac{a/b}{2} = \frac{a}{2b}$.
   Since a and $2b$ are integers and $2b \neq 0$, s is rational.
   Thus we have found a positive rational number s such that $s < r$.
   This contradicts the supposition that r is the *least* positive rational number.
   Therefore, there is no least positive rational number.

8. *Proof (by contraposition):* Suppose a and b are *[particular but arbitrarily chosen]* real numbers such that $a \geq 25$ and $b \geq 25$.
   Then $a + b \geq 25 + 25 = 50$.
   Hence if $a + b < 50$, then $a < 25$ or $b < 25$.

15. *Counterexample:* Let $a = 4$ and $n = 2$.
    Then $a|n^2$ because $4|4$ (since $4/4$ is an integer),
    but $a \nmid n$ because $4 \nmid 2$ (since $2/4$ is not an integer).

20. *Counterexample:* $\sqrt{2}$ is irrational.
    Also $\sqrt{2} - \sqrt{2} = 0$ and 0 is rational (by Theorem 3.2.1).
    Thus $\exists$ irrational numbers whose difference is rational.

Section 3.7

6. This statement is false.
   $\sqrt{2}/4 = (1/4)\cdot\sqrt{2}$, which is a product of a nonzero rational number and an irrational number.
   By exercise 18 of section 3.6, such a product is irrational.

12. *Proof:* Suppose not.
    That is, suppose $\exists$ an irrational number x such that $\sqrt{x}$ is rational.
    By definition of rational, $\sqrt{x} = a/b$ for some integers a and b with $b \neq 0$.
    Then $x = (\sqrt{x})^2 = (a/b)^2 = a^2/b^2$.
    But a^2 and b^2 are both integers (being products of integers), and $b \neq 0$ by the zero product property.
    Hence, x is rational (by definition of rational).
    This contradicts the supposition that x is irrational, and so the supposition is false.
    Therefore, the square root of an irrational number is irrational.

24. There are (at least) two possible ways of proving this result:

    Proof (by contradiction): Suppose not.
    Suppose there exist two distinct real numbers b_1 and b_2 such that for all real numbers r, (1) $b_1 r = r$ and (2) $b_2 r = r$.
    Then $b_1 b_2 = b_2$ (by (1) with $r = b_1$) and $b_2 b_1 = b_1$ (by (2) with $r = b_2$).
    Consequently, $b_2 = b_1 b_2 = b_2 b_1 = b_1$ by substitution and the commutative law of multiplication.
    But this implies that $b_1 = b_2$, which contradicts the supposition that b_1 and b_2 are distinct.
    [Thus the supposition is false and there exists at most one real number b such that $br = r$ for all real numbers r.]

    Proof (direct): Suppose b_1 and b_2 are real numbers such that (1) $b_1 r = r$ and
    (2) $b_2 r = r$ for all real numbers r.
    By (1) $b_1 b_2 = b_2$,
    and by the commutative law of multiplication and (2), $b_1 b_2 = b_2 b_1 = b_1$.
    Since both b_1 and b_2 are equal to $b_1 b_2$, we conclude that $b_1 = b_2$.

Section 3.8

3. b. $z = 4$

5. $e = 41/24$

17. *Proof:* Let a and b be any positive integers.
    Part 1 (proof that if $\gcd(a, b) = a$ *then* $a \mid b$*):*
    Suppose that $\gcd(a, b) = a$.
    By definition of greatest common divisor, $\gcd(a, b) \mid b$,
    and so by substitution, $a \mid b$.

Part 2 (proof that if $a \mid b$ *then* $\gcd(a,b) = a$ *):*
Suppose that $a \mid b$.
Then since it is also the case that $a \mid a$, a is common divisor of a and b.
Thus by definiton of greatest common divisor, $a \leq \gcd(a,b)$.
On the other hand, since no integer greater that a divides a, the greatest common divisor of a and b is less than or equal to a.
In symbols, $\gcd(a,b) \leq a$.
Therefore, since $a \leq \gcd(a,b)$ and $\gcd(a,b) \leq a$, then $\gcd(a,b) = a$.

22. a. *Proof:* Suppose a, d, q and r are integers such that $a = d \cdot q + r$ and $0 \leq r < d$.
[We must show that $q = \lfloor a/d \rfloor$ *and* $r = a - \lfloor a/d \rfloor \cdot d$*.]*
Solving $a = d \cdot q + r$ for r gives $r = a - d \cdot q$,
and substitutiong into $0 \leq r < d$ gives $0 \leq a - d \cdot q < d$.
Add $d \cdot q$: Then $d \cdot q \leq a < d + d \cdot q = d \cdot (q+1)$,
and so $q \leq a/d < q + 1$.
Thus by definition of floor, $q = \lfloor a/d \rfloor$,
and by substitution into $r = a - d \cdot q$, we have $r = a - \lfloor a/d \rfloor \cdot d$ *[as was to be shown]].*

b. $r := B$, $a := A$, $b : -B$

while $(b \neq 0)$
    $r := a - \lfloor a/b \rfloor \cdot b$
    $a := b$
    $b := r$
end while
$gcd := a$

24. b. $\text{lcm}(2 \cdot 3^2 \cdot 5, \quad 2^3 \cdot 3) = 2^3 \cdot 3^2 \cdot 5 = 360$
c. $\text{lcm}(3500, 1960) = \text{lcm}(2^2 \cdot 5^3 \cdot 7, \quad 2^3 \cdot 5 \cdot 7^2) = \quad 2^3 \cdot 5^3 \cdot 7^2 = 49{,}000$