CSE 20: Discrete Mathematics

Spring Quarter 1998

Solutions for Homework Assignment #3

Section 3.1

1. This problem is solved in the book.

3. This problem is solved in the book.

4. Let $a = 1$ and $b = 0$. Then $\sqrt{a+b} = \sqrt{1} = 1$ and $\sqrt{a} + \sqrt{b} = \sqrt{1} + \sqrt{0} = 1$ also. Hence $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ for these values of a and b. Since this shows the existence of numbers a and b such that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, we are done.

9. $1^2 - 1 + 11 = 11$, which is prime. $2^2 - 2 + 11 = 13$, which is prime.
   $3^2 - 3 + 11 = 17$, which is prime. $4^2 - 4 + 11 = 23$, which is prime.
   $5^2 - 5 + 11 = 31$, which is prime. $6^2 - 6 + 11 = 41$, which is prime.
   $7^2 - 7 + 11 = 53$, which is prime. $8^2 - 8 + 11 = 67$, which is prime.
   $9^2 - 9 + 11 = 83$, which is prime. $10^2 - 10 + 11 = 101$, which is prime.

11. This problem is solved in the book.

12. *Proof:* Suppose m and n are any *[particular but arbitrarily chosen]* odd integers. *[We must show that $m + n$ is even.]*

    By definition of odd, there exist integers r and s such that

    $$m = 2r + 1$$

    and

    $$n = 2s + 1.$$

    Then

    $$m + n = (2r + 1) + (2s + 1) = 2r + 2s + 2 = 2(r + s + 1).$$

    But $r + s + 1$ is an integer because r, s and 1 are integers and a sum of integers is a integer.

    Hence $m + n$ equals twice an integer, and so by definition of even, $m + n$ is even *[as was to be shown]*.

15. This problem is solved in the book.

18. *Start of proof:* Suppose x is any *[particular but arbitrarily chosen]* real number such that $x > 1$. *[We must show that $x^2 > x$.]*

23. This problem is solved in the book.

24. This incorrect proof begs the question. The second sentence of the "proof" states a conclusion that follows from the assumption that $m \cdot n$ is even. The next to last sentence states this conclusion as if it were known to be true. But it is not known to be true. In fact, it is the main task of the proof to derive this conclusion, not from the assumption that it is true but from the hypothesis of the theorem.

27. *Counterexample:* Let $m = 5$ and $n = 3$. Then m and n are odd but $m - n = 2$, which is even.

30. *Proof:* Let m and n be any even integers. By definition of even, $m = 2r$ and $n = 2s$ for some integers r and s. By substitution,

$$m - n = 2r - 2s = 2(r - s).$$

Since $r - s$ is an integer (being a difference of integers), $m - n$ equals twice some integer, and so $m - n$ is even by definition of even.

Section 3.2

2.

$$3.9602 = \frac{39602}{10000}$$

6. This problem is solved in the book.

9. This problem is solved in the book.

10. $2p + 3q$ and $5q$ are both integers because p and q are integers and products and sums of integers are integers. Also by the zero product property, $5q \neq 0$ since $5 \neq 0$ and $q \neq 0$.

Hence $2p + 3q/5q$ is a quotient of integers with a nonzero denominator, and so it is rational.

13. This problem is solved in the book.

15. *Proof:* Suppose r and s are any *[particular but arbitrarily chosen]* rational numbers. *[We must show that $r - s$ is rational.]*

By definition of rational, $r = a/b$ and $s = c/d$ for some integers a, b, c and d with $b \neq 0$ and $d \neq 0$. Then by substitution and the laws of algebra,

$$r - s = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

But $ad - bc$ and bd are both integers because a, b, c and d are integers and products and sums of integers are integers. Furthermore, bd is nonzero by the zero product property because $b \neq 0$ and $d \neq 0$.

Hence $r - s$ is is a quotient of integers with a nonzero denominator, and so it is rational *[as was to be shown]*.

21. This problem is solved in the book.

22. *Proof:* Suppose r is any rational number. Then $r^2 = r \cdot r$ is a product of rational numbers and hence is rational by exercise 13.

    Also 2, which is an integer, is rational by exercise 11. Thus, $2r^2$ is rational by exercise 13.

    By exercise 15, $2r^2 - r$ is rational (because it is a difference of two rational numbers).

    Finally, 1, which is an integer, is rational by exercise 11. So by Theorem 3.2.2, $2r^2 - r + 1 = (2r^2 - r) + 1$ is rational.

29. This incorrect proof just shows the theorem to be true in the one case where one of the rational numbers is $1/4$ and the other is $1/2$. A correct proof must show the theorem is true for *any* two rational numbers.

Section 3.3

7. This problem is solved in the book.

8. Yes, $6a \cdot 10b = 4(15ab)$ and $15ab$ is an integer because products of integers are integers.

15. *Proof:* Suppose a, b and c are integers and $a|b$ and $a|c$. *[We must show that $a|(b-c)$.]*

    By definition of divisibility, there exist integers r and s such that $b = ar$ and $c = as$. Then

    $$b - c = ar - as = a(r - s)$$

    by substitution and the distributive law. But $r - s$ is an integer since it is a difference of two integers. Hence $a|(b - c)$ *[as was to be shown]*.

20. *Proof:* Suppose n is any integer. By basic algebra,

    $$n(6n + 3) = 3[n(2n + 1)]$$

    But $n(2n+1)$ is an integer because n is an integer and sums and products of integers are integers.

    Hence by definition of divisibility, $n(6n + 3)$ is divisible by 3.

24. *Counterexample:* Let $a = 2$, $b = 3$ and $c = 1$. Then $a|(b + c)$ because $2|4$ but $a \nmid b$ because $2 \nmid 3$.

28. Let n be the number of minutes past 4 p.m. when the athletes first return to the start together. Then n is the smallest multiple of 8 that is also a multiple of 10. this number is 40.

    Hence the first time the athletes will return to the start together will be 4:40 p.m.

35. If the standard factored form for for a is

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$$

then

$$\begin{aligned} a^3 &= \left(p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}\right)^3 \\ &= \left(p_1^{e_1}\right)^3 \cdot \left(p_2^{e_2}\right)^3 \cdot \ldots \cdot \left(p_k^{e_k}\right)^3 \\ &= p_1^{3e_1} \cdot p_2^{3e_2} \cdot \ldots \cdot p_k^{3e_k} \end{aligned}$$

which is the standard factored form for for a^3.

38. *Proof:* Suppose n is a nonnegative integer whose decimal representation ends in a 5. By definition, this means

$$n = d_k \cdot 10^k + \ldots + d_2 \cdot 10^2 + d_1 \cdot 10 + 5$$

for some integer $k \geq 0$. By factoring out a 5, we get

$$n = 5\big(d_k \cdot 2 \cdot 10^{(k-1)} + \ldots + d_2 \cdot 2 \cdot 10^1 + d_1 \cdot 2 + 1\big)$$

where

$$d_k \cdot 2 \cdot 10^{(k-1)} + \ldots + d_2 \cdot 2 \cdot 10^1 + d_1 \cdot 2 + 1$$

is an integer because sums and products of integers are integers.

Hence n is divisible by 5.

Section 3.3

4. $q = 0$, $d = 5$

9. It is $61 = 2 \cdot 30 + 1$. Hence the results are

   a. 30

   b. 1

14. The number of days from January 1, 1990 to January 1, 2000 is ten years times 365 days per year plus 2 leap year days (in 1992 and 1996), which gives a total of 3652 days.

    Now $3652 \; mod \; 7 = 5$ and $3652 \; div \; 7 = 521$, so January 1, 2000 is 521 weeks and 5 days from January 1, 1990. As January 1, 1990is a Monday, January 1, 2000 must be a Saturday.

16. a. This problem is solved in the book.

    b. a_{ij} is stored in location $8304 + 4(i-1) + (j-1)$. Thus,

$$n = 4(i-1) + (j-1).$$

c. $r = n \; div \; 4 + 1$ and $s = n \; mod \; 4 + 1$.

Note that by using the formulas

$$a \; mod \; b = a - \left\lfloor \frac{a}{b} \right\rfloor b \qquad \text{and} \qquad a \; div \; b = \left\lfloor \frac{a}{b} \right\rfloor$$

this answer can be checked for consistency with the result of part b:

$$\begin{aligned} 4(r-1) + (s-1) &= 4\,[(n \; div \; 4 + 1) - 1] + [(n \; mod \; 4 + 1) - 1] \\ &= 4 \cdot \left\lfloor \frac{n}{4} \right\rfloor + (n - \left\lfloor \frac{n}{4} \right\rfloor \cdot 4) \\ &= n \end{aligned}$$

20. *Proof:* Suppose n is any integer. By the quotient-remainder theorem with $d = 3$, there exist integers q and r such that $n = 3q + r$ and $0 \le r < 3$. But the only nonnegative integers that are less than 3 are 0, 1, and 2.

    Therefore, $n = 3q + 0 = 3q$, or $n = 3q + 1$, or $n = 3q + 2$ for some integer q.

21. a. This problem is solved in the book.

    b. For any integer n it is

$$n(n+1)(n+2) \; mod \; 3 \; = \; 0.$$

22. *Proof:* Suppose n is any integer. *[We must show that $n^2 = 3k$ or $n^2 = 3k + 1$ for some integer k.]* By exercise 20, $n = 3q$, or $n = 3q + 1$, or $n = 3q + 2$ for some integer q.

    Case1 ($n = 3q$ for some integer q): In this case,

$$n^2 = (3q)^2 = 3(3q^2)$$

    Let $k = 3q^2$. Then k is an integer because it is a product of integers. Hence $n^2 = 3k$ for some integer k.

    Case2 ($n = 3q + 1$ for some integer q): In this case,

$$n^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$$

    Let $k = 3q^2 + 2q$. Then k is an integer because it is a sum of products of integers. Hence $n^2 = 3k + 1$ for some integer k.

    Case3 ($n = 3q + 2$ for some integer q): In this case,

$$n^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1 = 3(3q^2 + 4q + 1) + 1$$

    Let $k = 3q^2 + 4q + 1$. Then k is an integer because it is a sum of products of integers. Hence $n^2 = 3k + 1$ for some integer k.

    In all tree cases, either $n^2 = 3k$ or $n^2 = 3k + 1$ for some integer k. This is what was to be shown.

25. *Proof:* Suppose n is any integer. *[We must show that $n^2 = 4k$ or $n^2 = 4k + 1$ for some integer k.]*

By the quotient-remainder theorem $n = 4q$, or $n = 4q + 1$, or $n = 4q + 2$, or $n = 4q + 3$ for some integer q.

Case1 ($n = 4q$ for some integer q): In this case,

$$n^2 = (4q)^2 = 4(4q^2)$$

Let $k = 4q^2$. Then k is an integer because it is a product of integers. Hence $n^2 = 4k$ for some integer k.

Case2 ($n = 4q + 1$ for some integer q): In this case,

$$n^2 = (4q + 1)^2 = 16q^2 + 8q + 1 = 4(4q^2 + 2q) + 1$$

Let $k = 4q^2 + 2q$. Then k is an integer because it is a sum of products of integers. Hence $n^2 = 4k + 1$ for some integer k.

Case3 ($n = 4q + 2$ for some integer q): In this case,

$$n^2 = (4q + 2)^2 = 16q^2 + 16q + 4 = 4(4q^2 + 4q + 1)$$

Let $k = 4q^2 + 4q + 1$. Then k is an integer because it is a sum of products of integers. Hence $n^2 = 4k$ for some integer k.

Case4 ($n = 4q + 3$ for some integer q): In this case,

$$n^2 = (4q + 3)^2 = 16q^2 + 24q + 9 = 16q^2 + 24q + 8 + 1 = 4(4q^2 + 6q + 2) + 1$$

Let $k = 4q^2 + 6q + 2$. Then k is an integer because it is a sum of products of integers. Hence $n^2 = 4k + 1$ for some integer k.

It follows that in all four possible cases, $n^2 = 4k$ or $n^2 = 4k + 1$ for some integer k *[as was to be shown]*.

36. a. *Proof:* Let x and y be any real numbers.

Case 1 (x and y are both nonnegative): In this case, $|x| = x$, $|y| = y$ and xy is also nonnegative. So

$$|xy| = xy = |x| \cdot |y|.$$

Case 2 (x is nonnegative and y is negative): In this case, $|x| = x$, $|y| = -y$ and $xy \le 0$. So

$$|xy| = -xy = x(-y) = |x| \cdot |y|.$$

Case 3 (x is negative and y is nonnegative): In this case, $|x| = -x$, $|y| = y$ and $xy \le 0$. So

$$|xy| = -xy = (-x)y = |x| \cdot |y|.$$

Case 4 (x and y are both negative): In this case, $|x| = -x$, $|y| = -y$ and xy is nonnegative. So

$$|xy| = xy = (-x)(-y) = |x| \cdot |y|.$$

Therefore in all four possible cases, $|xy| = |x| \cdot |y|$ *[as was to be shown].*

b. This problem is solved in the book.

c. *Proof:* Let c be any positive real number and x be any real number.

Part 1 (Proof that if $-c \leq x \leq c$ then $|x| \leq c$):
Suppose that

$$-c \leq x \leq c \tag{1}$$

By the trichotomy law (see Appendix A, T16), either

$$x \geq 0 \qquad \text{or} \qquad x < 0.$$

Case 1 ($x \geq 0$): In this case, $|x| = x$ and so by substitution into (1), $-c \leq |x| \leq c$. In particular, $|x| \leq c$.

Case 2 ($x < 0$): In this case, $|x| = -x$ and so $-|x| = x$. Hence by substitution into (1), $-c \leq -|x| \leq c$. In particular, $-c \leq -|x|$. Multiplying both sides by -1 gives $c \geq |x|$, or equivalently, $|x| \leq c$.

Therefore, regardless of whether $x \geq 0$ or $x < 0$, $|x| \leq c$ *[as was to be shown].*

Part 2 (Proof that if $|x| \leq c$ then $-c \leq x \leq c$):
Suppose that

$$|x| \leq c \tag{2}$$

By the trichotomy law (see Appendix A, T16), either

$$x \geq 0 \qquad \text{or} \qquad x < 0.$$

Case 1 ($x \geq 0$): In this case, $|x| = x$ and so by substitution into (2), $x \leq c$. Since $x \geq 0$ and $0 \geq -c$, by transitivity of order (Appendix A, T17), $x \geq -c$. Hence $-c \leq x \leq c$.

Case 2 ($x < 0$): In this case, $|x| = -x$ and so by substitution into (2), $-x \leq c$. Multiplying both sides by -1 gives $x \geq -c$. Also since $x < 0$ and $0 < c$, $x \leq c$. Thus $-c \leq x \leq c$.

Therefore, regardless of whether $x \geq 0$ or $x < 0$, we conclude that $-c \leq x \leq c$ *[as was to be shown].*

d. *Proof:* Let any real numbers x and y be given. By part b.,

$$-|x| \le x \le |x|$$

and

$$-|y| \le y \le |y|$$

Hence, by the order properties of real numbers,

$$(-|x|) + (-|y|) \le x + y \le |x| + |y|$$

or equivalently

$$-(|x| + |y|) \le x + y \le |x| + |y|.$$

It follows immediately from part c. that $|x + y| \le |x| + |y|$.

Section 3.5

4.

$$\lfloor -57/2 \rfloor = \lfloor -28.5 \rfloor = -29$$
$$\lceil -57/2 \rceil = \lceil -28.5 \rceil = -28$$

12. The solution is in the book.

15. There are (at least) two possible ways of proving this result:

Proof 1: Suppose x is any *[particular but arbitrarily chosen]* real number.
Then $\lfloor x - 1 \rfloor$ is some integer: say $\lfloor x - 1 \rfloor = n$.
By definition of floor, $n \le x - 1 < n + 1$.
Adding 1 to all parts of this inequality gives $n + 1 \le x < n + 2$,
and thus by definition of floor $\lfloor x \rfloor = n + 1$.
Solving this equation for n gives $n = \lfloor x \rfloor - 1$.
But $n = \lfloor x - 1 \rfloor$ also (by definition on n).
Hence $\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1$ *[as was to be shown]*.

Proof 2: Suppose x is any *[particular but arbitrarily chosen]* real number.
Apply Theorem 3.5.1 with $m = -1$. Then

$$\lfloor x + m \rfloor = \lfloor x + (-1) \rfloor = \lfloor x - 1 \rfloor$$
$$\lfloor x \rfloor + m = \lfloor x \rfloor + (-1) = \lfloor x \rfloor - 1,$$

and so $\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1$ *[as was to be shown]*.

16. *Counterexample:* Let $x = 3/2$.
Then $\lfloor x^2 \rfloor = \lfloor (3/2)^2 \rfloor = \lfloor 9/4 \rfloor = 2$,
whereas $\lfloor x \rfloor^2 = \lfloor (3/2) \rfloor^2 = 1^2 = 1 \ne 2$.

22. *Counterexample:* Let $x = y = 1.9$.
Then $\lceil xy \rceil = \lceil 1.9 \cdot 1.9 \rceil = \lceil 3.61 \rceil = 4$,
whereas $\lceil x \rceil \cdot \lfloor y \rfloor = \lceil 1.9 \rceil \cdot \lfloor 1.9 \rfloor = 2 \cdot 1 = 2 \ne 4$.

23. The solution is in the book.

28. *Proof:* Let n be an odd integer. *[We must show that* $\lfloor \frac{n^2}{4} \rfloor = (\frac{n-1}{2})(\frac{n+1}{2})$.*]*
By definiton of odd, $n = 2k+1$ for some integer k.
Substituting into the left-hand side of the equation to be proved gives

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(2k+1)^2}{4} \right\rfloor = \left\lfloor \frac{4k^2+4k+1}{4} \right\rfloor = \left\lfloor k^2+k+\frac{1}{4} \right\rfloor = k^2+k.$$

where $\lfloor k^2+k+\frac{1}{4} \rfloor = k^2+k$ by definition of floor because k^2+k is an integer and $k^2+k < k^2+k+\frac{1}{4} < k^2+k+1$.
On the other hand, substituting into the right-hand side of the equation to be proved gives

$$\begin{aligned}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) &= \left(\frac{(2k+1)-1}{2}\right)\left(\frac{(2k+1)+1}{2}\right) \\ &= \left(\frac{2k}{2}\right)\left(\frac{2k+2}{2}\right) = k(k+1) = k^2+k\end{aligned}$$

also.
Thus the left- and right-hand side of the equation to be proved both equal k^2+k, and so the two sides are equal to each other.
In other words, $\lfloor \frac{n^2}{4} \rfloor = (\frac{n-1}{2})(\frac{n+1}{2})$ *[as was to be shown].*

30. The solution is in the book.