CSE 21 Midterm 2 Solutions

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July 21, 2005

1

The probability that a normal random variable X with mean $\mu = 68$ and standard deviation $\sigma = 2$ is between 66 and 71 is

$$\begin{aligned} Pr(66 \leq X \leq 71) &= Pr(1 \leq \frac{X-\mu}{\sigma} \leq 1.5) \\ &= Pr(0 \leq Z \leq 1) + Pr(0 \leq Z \leq 1.5) \\ &\approx .3413 + .4332 = .7745 \end{aligned}$$

where Z is a standard normal random variable.

2

Let X be uniformly random over $\{1, \ldots, 6\}$ and let Y be 1 if X is odd and 3 if X is even. Then

$$\begin{aligned} E(Y) &= 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 2 \\ \text{Var}(Y) &= E(Y^2) - E(Y)^2 = 1^2 \cdot \frac{1}{2} + 3^2 \cdot \frac{1}{2} - 2^2 = 1 \\ \text{StdDev}(Y) &= \sqrt{\text{Var}(Y)} = 1. \end{aligned}$$

3

If X has mean $\mu = 25$ and standard deviation $\sigma = 2$, then we can lower bound the probability that $X < 35$ as follows:

$$\begin{aligned} Pr(X < 35) &= 1 - Pr(X \geq 35) \\ &= 1 - Pr(X - \mu \geq 10) \\ &\geq 1 - Pr(|X - \mu| \geq 10) \\ &\geq 1 - \frac{\sigma^2}{10^2} && \text{by Chebyshev's inequality} \\ &= \frac{24}{25}. \end{aligned}$$

Since we don't know whether the distribution of X is symmetric about the mean we cannot assert that $Pr(X - \mu \geq 10) = \frac{1}{2}Pr(|X - \mu| \geq 10)$ to get a greater lower bound.

Also note that we cannot assert any nontrivial upper bound on $Pr(X < 35)$ since if X is the random variable that takes on each of the values $23, 27$ with probability $\frac{1}{2}$ then $\mu = 25, \sigma = 2$ and yet $Pr(X < 35) = 1$.

4

To count the number of ways a_n to pair off $2n$ people for tennis matches we need to decide what we mean by a *way*.

If we mean "how many ordered sequences $p_1, \ldots, p_n$ are there where each p_i is a subset of $\underline{2n}$ of size 2 and $\cup_i p_i = \underline{2n}$", then we have

$$\begin{aligned} a_n &= \begin{cases} \binom{2n}{2} a_{n-1} & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases} \\ &= \frac{(2n)!}{2^n}. \end{aligned}$$

On the other hand, if we mean "how many unordered sequences $p_1, \ldots, p_n$ are there where each p_i is a subset of $\underline{2n}$ of size 2 and $\cup_i p_i = \underline{2n}$, then we have

$$\begin{aligned} a_n &= \begin{cases} (2n-1)a_{n-1} & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases} \\ &= \frac{(2n)!}{2^n n!}. \end{aligned}$$

Notice that these 2 answers only differ by a factor of $n!$.

5

Suppose $a_n = a_{n-1} + 6a_{n-2} + 2^n, a_0 = 1, a_1 = 9$. To find a particular solution p_n to the inhomogeneous equation $p_n = p_{n-1} + 6p_{n-2} + 2^n$, we guess that p_n has the form $C2^n$ and solve for C. Then taking $n = 2$, C must satisfy $4C = 2C + 6C + 4$ and so $C = -1$ and $p_n = -2^n$.

Next, the homogeneous equation $h_n = h_{n-1} + 6h_{n-2}$ has a corresponding characteristic polynomial equation $x^2 = x + 6$, which has roots $-2, 3$, from the quadratic formula. So h_n must be of the form $A(-2)^n + B3^n$.

Combining these, a_n must have the form $A(-2)^n + B3^n - 2^n$. By using the initial conditions, we find that $A = -1, B = 3$. So $a_n = -(-2)^n + 3^{n+1} - 2^n$.