CSE 21 Midterm Solutions

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1

To count the number of arrangements of 5 boys and 4 girls in which every girl is next to at least 1 other girl, note that such an arrangement must either have a run of 4 girls or 2 runs of 2 girls each.

So the number of arrangements of 5 indistinguishable boys and 4 indistinguishable girls with each girl next to at least 1 other girl is the number of strings of the form $b^{x_1}g^2b^{x_2}g^2b^{x_3}$ of size $4+5=9$, which is the same as the number of integer solutions to $x_1+x_2+x_3=5, x_i \geq 0$, and this is $\binom{7}{2}$.

The number of arrangements of 5 indistinguishable boys and 4 indistinguishable girls is $\binom{9}{4}$, so the probability that every girl is next to at least 1 other girl is $\frac{\binom{7}{2}}{\binom{9}{4}} = \frac{1}{6}$.

2

The number of arrangements of the letters in BOOKKEEPING in which the Ks are not consecutive but the Es are is the number of arrangements where the Es are consecutive minus the number of arrangements in which both the Ks and the Es are consecutive, and this is $\frac{10!}{2!^2} - \frac{9!}{2!} = 725760$.

3

The number of ways to distribute 15 identical donuts to 10 distinct people so that everyone gets at least 1 donut and the first 2 people get the same number of donuts is the same as the number of integer solutions to $2x_2+x_3+\cdots+x_{10} = 15, x_i \geq 1$. The number of such solutions in which $x_2 = j$ is $\binom{15-2j-1}{8-1}$. So the total number of such solutions is $\sum_{j=1}^{3}\binom{15-2j-1}{8-1} = 920$.

4

a

One sample space that can model when a pair of dice is thrown is $S = [6]^2 = \{(x, y) \mid x, y \in \{1, \ldots, 6\}\}$, although any space with ≥ 36 points will do. The probability measure on S is uniform: $\forall x \in S \; Pr(\{x\}) = \frac{1}{|S|} = \frac{1}{36}$.

If one does not need to distinguish the dice then it is possible to get away with a smaller sample space, say of size 21, but this is a poor model since then (1) the probability measure is nonuniform, making analysis cumbersome, (2) the fact that the dice are independent of each other is obscured, and (3) if one later cares about questions such as "what is the probability that the first die is 3?" then this sample space becomes inappropriate because there is no such event in it.

b

Let X_i be the value of die i for $i = 1, 2$. Then

$$\begin{aligned} Pr(X_1 X_2 > 9) &= Pr(X_1 = 2 \wedge X_2 \geq 5) \\ &\quad + Pr(X_1 = 3 \wedge X_2 \geq 4) \\ &\quad + Pr(X_1 = 4 \wedge X_2 \geq 3) \\ &\quad + Pr(X_1 = 5 \wedge X_2 \geq 2) \\ &\quad + Pr(X_1 = 6 \wedge X_2 \geq 2) \\ &= \frac{1}{6^2}(2 + 3 + 4 + 5 + 5) = \frac{19}{36}. \end{aligned}$$

5

The probability that the 4 cards after the first 3 do not have any diamonds given that the first 3 are diamonds is $\frac{\binom{39}{4}}{\binom{49}{4}} \approx .39$, since the 4 cards must be selected from the 39 remaining non-diamond cards.

6

Suppose 40 machines contain 6 defectives and 10 distinct machines are chosen at random. The number of those chosen that are defective is a hypergeometric random variable. The probability that this is 3 is $\frac{\binom{6}{3}\binom{34}{7}}{\binom{40}{10}} \approx .13$, since this is (the number of ways of choosing 3 defectives out of 6) times (the number of ways of choosing 7 nondefectives out of 34) divided by (the number of ways of choosing 10 out of 40).

7

Let

- U be a set of students
- L be those taking a language class
- S be those taking a science class
- M be those taking a math class

and suppose that

$$\begin{aligned} |U| &= 100 & |L \cap M| &= 13 \\ |L| &= 33 & |L \cap S| &= 14 \\ |S| &= 28 & |M \cap S| &= 15 \\ |M| &= 29 & |L \cap S \cap M| &= 5. \end{aligned}$$

Then

$$\begin{aligned} |U - L \cup S \cup M| =& |U| - |L| - |S| - |M| \\ &+ |L \cap S| + |L \cap M| + |S \cap M| \\ &- |L \cap S \cap M| \\ =& 100 - 33 - 28 - 29 + 14 + 13 + 15 - 5 = 47. \end{aligned}$$