CSE 21 Homework 8 Solutions

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Exercises from Applied Combinatorics

6.1.7

The number of 4-tuples $(x_1, \ldots, x_4)$ of integers with each $x_i \geq 3$ and with $\sum x_i = 18$ is the coefficient of x^{18} in the power series $(\sum_{i=3}^{\infty} x^i)^4$.

6.1.17

The generating function for the number of selections of r objects from 8 types of objects such that the number of objects of each type is a multiple of 5 is $(\sum_{i=0}^{\infty} x^{5i})^8$. The idea here is that the x^i term of the jth factor represents choosing i objects of type j.

6.1.27

The generating function for the number of ways 8 people can each pick 2 fruits of different types from a bowl containing an infinite number of apples, oranges, bananas is $(xy + xz + yz)^8$. Here the $x^i y^j z^k$ term in factor l represents that person l chose i apples, j oranges, and k bananas.

6.2.11

To simplify the following problems, let us use the notation $\operatorname{coef}(f(x), g(x))$ where f is a monomial to represent the coefficient of f in the power series of g. Notice that coef is linear in the second argument and inverse linear in the first (over the complex field, not the polynomial ring), that $\operatorname{coef}(x^i, (1-x)^{-n}) = \binom{i+n-1}{n-1}$, and that $\operatorname{coef}(f(x), g(x)h(x)) = \operatorname{coef}(f(x)h(x)^{-1}, g(x))$ where h is a monomial. Also note the following substitution rule: $\operatorname{coef}(f \circ h(x), g \circ h(x)) = \operatorname{coef}(f(x), g(x))$ for h a monomial.

a

$$\begin{aligned}\operatorname{coef}(x^{12}, x^2(1-x)^{-10}) &= \operatorname{coef}(x^{10}, (1-x)^{-10}) \\ &= \binom{10+10-1}{10-1} = 92378.\end{aligned}$$

b

$$\begin{aligned}\operatorname{coef}(x^{12}, \frac{x^2-3x}{(1-x)^4}) &= \operatorname{coef}(x^{10}, (1-x)^{-4}) - 3\operatorname{coef}(x^{11}, (1-x)^{-4}) \\ &= \binom{10+4-1}{4-1} - 3\binom{11+4-1}{4-1} = -806.\end{aligned}$$

c

$$\begin{aligned}\operatorname{coef}(x^{12}, \frac{(1-x^2)^5}{(1-x)^5}) &= \operatorname{coef}(x^{12}, \frac{(1-x)^5(1+x)^5}{(1-x)^5}) \\ &= \operatorname{coef}(x^{12}, (1+x)^5) = 0.\end{aligned}$$

d

$$\begin{aligned}\operatorname{coef}(x^{12}, \frac{x+3}{1-2x+x^2}) &= \operatorname{coef}(x^{12}, \frac{x+3}{(1-x)^2}) \\ &= \operatorname{coef}(x^{11}, (1-x)^{-2}) + 3\operatorname{coef}(x^{12}, (1-x)^{-2}) \\ &= \binom{11+2-1}{2-1} + 3\binom{12+2-1}{2-1} = 51.\end{aligned}$$

e

$$\begin{aligned}\operatorname{coef}(x^{12}, \frac{b^m x^m}{(1-bx)^{m+1}}) &= b^{12}\operatorname{coef}((bx)^{12}, \frac{(bx)^m}{(1-bx)^{m+1}}) \\ &= b^{12}\operatorname{coef}(x^{12}, \frac{x^m}{(1-x)^{m+1}}) \qquad \text{by substitution} \\ &= b^{12}\binom{12}{m}.\end{aligned}$$

6.2.15

a

$$\operatorname{coef}(x^{12}, (1-x)^8) = 0.$$

b

$$\begin{aligned}\operatorname{coef}(x^{12},(1+x)^{-1}) &= (-1)^{12}\operatorname{coef}((-x)^{12},(1-(-x))^{-1}) \\ &= (-1)^{12}\operatorname{coef}(x^{12},(1-x)^{-1}) \qquad \text{by substitution} \\ &= (-1)^{12} = 1.\end{aligned}$$

c

$$\begin{aligned}\operatorname{coef}(x^{12},(1+x)^{-8}) &= (-1)^{12}\operatorname{coef}((-x)^{12},(1-(-x))^{-8}) \\ &= (-1)^{12}\operatorname{coef}(x^{12},(1-x)^{-8}) \qquad \text{by substitution} \\ &= (-1)^{12}\binom{12+8-1}{8-1} \\ &= \binom{19}{7} = 50388.\end{aligned}$$

d

$$\begin{aligned}\operatorname{coef}(x^{12},(1-4x)^{-5}) &= 4^{12}\operatorname{coef}((4x)^{12},(1-4x)^{-5}) \\ &= 4^{12}\operatorname{coef}(x^{12},(1-x)^{-5}) \qquad \text{by substitution} \\ &= 4^{12}\binom{12+5-1}{5-1} \approx 3\cdot 10^{10}.\end{aligned}$$

e

$$\begin{aligned}\operatorname{coef}(x^{12},(1+x^3)^{-4}) &= \operatorname{coef}(x^4,(1+x)^{-4}) \\ &= (-1)^4\operatorname{coef}((-x)^4,(1-(-x))^{-4}) \\ &= (-1)^4\operatorname{coef}(x^4,(1-x)^{-4}) \\ &= (-1)^4\binom{4+4-1}{4-1} = 35.\end{aligned}$$

6.2.29

The number of distributions of 10 identical steaks and 15 identical lobsters to 4 people such that each person gets at most 5 steaks and at most 5 lobsters is the product of the number N_1 of integer solutions to the system $x_1+\cdots+x_4 = 10, 0 \le x_i \le 5$ and the number N_2 of integer solutions to the system $y_1+\cdots+y_4 = 15, 0 \le y_i \le 5$. We solve each by using inclusion-exclusion:

$$N_1 = \binom{10+4-1}{4-1} - \binom{4}{1}\binom{10-6+4-1}{4-1} = 146$$

$$N_2 = \binom{15+4-1}{4-1} - \binom{4}{1}\binom{15-6+4-1}{4-1} + \binom{4}{2}\binom{15-12+4-1}{4-1} = 56.$$

The product is $N_1 \cdot N_2 = 8176$. Note that the answer in the book is incorrect, since it does not parse correctly.

6.2.43

$$\begin{aligned} a_r &= |\{(x_1, \ldots, x_i) \mid i \in \underline{n}, x_j \in \underline{n}, \sum x_j = r\}| \\ &= \sum_{i=1}^{n} |\{(x_1, \ldots, x_i) \mid x_j \in \underline{n}, \sum x_j = r\}| \\ &= \sum_{i=1}^{n} \mathrm{coef}(x^r, (\sum_{j=1}^{n} x^j)^i). \end{aligned}$$

Now take $n = 6$.

6.3.3

$$\begin{aligned} &|\{(x_1, \ldots, x_r) \mid 1 \le x_1 \le \cdots \le x_r, \sum x_i = r, \forall j \in \mathbb{N}\ |\{i \in \underline{r} \mid x_i = j\}| \le 3\}| \\ &= \text{number of integer solutions to } (1y_1 + 2y_2 + \cdots + ry_r = r, 0 \le y_i \le 3) \\ &= \mathrm{coef}(x^r, \prod_{i=1}^{\infty}(1 + x^i + x^{2i} + x^{3i})). \end{aligned}$$

6.3.5

The generating function for the number of ways to make r cents' change from coins of the denominations $d_1, \ldots, d_k$ is the product from $i = 1$ to k of the generating function for the number of ways to make r cents' change from the denomination d_i alone. For $(d_1, \ldots, d_k) = (1, 5, 10, 25)$, this is $(1 + x + x^2 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^{10} + x^{20} + \cdots)(1 + x^{25} + x^{50} + \cdots) = (1 - x)^{-1}(1 - x^5)^{-1}(1 - x^{10})^{-1}(1 - x^{25})^{-1}$.

How to interpret exponential generating functions

Suppose we have a selection process where we have k types of objects and k "choices". Choice i is to select an element x_i of the set S_i (we may think of S_i as the ith sample space) and this yields $X_i = f_i(x_i)$ objects of type i (we may think of X_i as an integer random variable on the space S_i). Once all k choices are made (equivalent to selecting a single point from the product space $\prod S_i$) and all objects are yielded, we arrange the objects of the k types in a row.

The number of choice, arrangement pairs yielding a total of n objects is

$$e_n =_{\mathrm{def}} |\{(x, y) \mid x \in \prod S_i, y \in \underline{k}^{\underline{n}}, \forall i\ |y^{-1}(i)| = f_i(x_i)\}|,$$

the exponential generating function of which is

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Big(\sum_{\substack{j_1,\ldots,j_k\geq 0\\ \sum j_i=n}}\binom{n}{j_1,\ldots,j_k}\prod_{i=1}^{k}|f_i^{-1}(j_i)|\Big)\frac{x^n}{n!}\\
=&\sum_{n=0}^{\infty}\Big(\sum_{\substack{j_1,\ldots,j_k\geq 0\\ \sum j_i=n}}\prod_{i=1}^{k}\frac{|f_i^{-1}(j_i)|}{j_i!}\Big)x^n \qquad (1)\\
=&\prod_{i=1}^{k}\sum_{j=0}^{\infty}|f_i^{-1}(j)|\frac{x^j}{j!},
\end{aligned}$$

which is the product over $i \in \underline{k}$ of the exponential generating function of $|f_i^{-1}(j)|$, the number of ways of yielding j objects of type i.

We can also interpret this probabilistically: if we select X_i objects of type i and the X_i are independent, then

$$\prod_{i=1}^{k}E\Big(\frac{t^{X_i}}{X_i!}\Big)=E\Big(\prod_{i=1}^{k}\frac{t^{X_i}}{X_i!}\Big)=E\Big(\frac{t^{\sum X_i}}{\prod X_i!}\Big).$$

6.4.5

Let us use the above interpretation to find an exponential generating function for the number of sequences of length n of cards from a deck of 52 cards where the suits have been erased.

$k = 13$ since there are 13 kinds of cards. S_i has size 5 and $f_i : S_i \to \{0,\ldots,4\}$ is a bijection (which one does not matter). The LHS of equation (1) is the desired generator and so this must be the RHS, namely $(1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!})^{13}$.

6.4.9

a

The number of 10 letter words formed from the alphabet of 26 letters with each of e, n, r, s occurring at most once is the 10th coefficient of the exponential generating function $(1+\frac{x^1}{1!})^4(1+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots)^{26-4}=(1+x)^4e^{22x}$.

To see this, we use the above interpretation letting $k = 26$, $S_1,\ldots,S_4 = \{0,1\}, \forall i \in [5,26]\ S_i = \mathbb{N}$, each f_i is the identity.

But since this doesn't seem to help compute the 10th coefficient (any more than the below argument anyway), let's return to more mundane combinatorics. We can first choose some i of the letters e, n, r, s to appear, then choose an i-permutation of the 10 locations for them to appear in, then choose the remaining $10-i$ letters. This gives a total of $\sum_{i=0}^{4}\binom{4}{i}P(10,i)(26-4)^{10-i}\approx 1.12\cdot 10^{14}$.

b

The number of 10 letter words formed from the alphabet of 26 letters with each of e, n, r, s occurring at least once is the 10th coefficient of the exponential generating function $(e^x - 1)^4 e^{22x}$, for similar reasons.

Again we could expand this, calculate the individual coefficients, and add them up, but let us use a simpler argument. The desired number is $\sum_{i=0}^{4} (-1)^i \binom{4}{i} (26-i)^{10} \approx 9.7 \cdot 10^{11}$ from simple inclusion-exclusion.

6.4.10

a

The number of r-digit 3-ary sequences in which no digit occurs exactly twice can be computed from inclusion-exclusion: $3^r - \binom{3}{1}\binom{r}{2}2^{r-2} + \binom{3}{2}\binom{r}{2,2,r-4}1^{r-4} - \binom{3}{3}\binom{r}{2,2,2}$, where the last multinomial is 0 unless $r = 6$.

b

The number of r-digit 3-ary sequences in which 0 and 1 each occur a positive even number of times is the rth coefficient of the exponential generating function

$$\begin{aligned}
&\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right) \\
=&\left(\frac{e^x + e^{-x}}{2} - 1\right)^2 e^x \\
=&\left(\frac{1}{4}e^{2x} - \frac{1}{2}e^x + 2 - \frac{1}{2}e^{-x} + \frac{1}{4}e^{-2x}\right) e^x \\
=&\frac{1}{4}e^{3x} - \frac{1}{2}e^{2x} + 2e^x - \frac{1}{2} + \frac{1}{4}e^{-x},
\end{aligned}$$

which is

$$r!\left(\frac{1}{4}\begin{cases} \frac{1}{(\frac{r}{3})!} & \text{if } 3 \mid r \\ 0 & \text{else} \end{cases} + \frac{1}{2}\begin{cases} \frac{1}{(\frac{r}{2})!} & \text{if } 2 \mid r \\ 0 & \text{else} \end{cases} + 2\frac{1}{r!} - \frac{1}{2}\begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{else} \end{cases} + \frac{1}{4}\frac{(-1)^r}{r!}\right).$$