# CSE 21 Homework 7 Solutions

Chris Calabro

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## Exercises from Applied Combinatorics

### 7.1.7

Suppose that at time step 1 there is 1 male and 1 female rabbit who were both just born and that at each successive time step the male rabbits of age strictly larger than 1 time step arbitrarily match up with the female rabbits of age strictly larger than 1 time step and each such pairing produces another male,female pair of rabbits for this time step. Then the number of pairs of rabbits at time $n$ is $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ and $a_0 = a_1 = 1$, since this is the number of pairs at the previous time step plus those strictly older than 1 time step.

### 7.1.11

This problem is easy to answer but hard to state because any description of the rules for how the light moves will be verbose. We will do so by first describing a crude model for the physics of the problem, then by giving a much more concise combinatoric model that discards the irrelevant physical details, and finally by giving a recurrence relation that discards the irrelevant set-theoretic details.

Let $l_1, l_2, l_3$ be 3 horizontal lines in the plane with $l_1$ above $l_2$ above $l_3$. Let $p_1, p_2$ be 2 points above $l_1$ so that the ray from $p_1$ to $p_2$ intersects $l_1$ but not perpendicularly. We will think of $p_1$ as a light source and $p_2$ as giving a direction. Say an *internal reflection path* is a piecewise linear curve with $p_1$ as the only endpoint, crossing through $p_2$, with cusps only at $l_1, l_2, l_3$, the "angle of incidence" equals the "angle of reflection" at each cusp, the first cusp (closest to $p_1$) is not at $l_1$, the curve is infinitely long.

For fixed $p_1, p_2, l_1, l_2, l_3$ and $n \in \mathbb{N}$, we want to know the number $a_n$ of internal reflection paths with exactly $n$ cusps. The notions of straight, horizontal, direction, perpendicularity, cusp, etc. are all irrelevant: $a_n$ is the number of sequences $x_1, \ldots, x_n$ with each $x_i \in \underline{3}$, $x_1 \neq 1$, and $\operatorname{sgn}(x_{i+1} - x_i) = (-1)^i$ for $i = 1, \ldots, n-1$. The idea here is that $x_i$ is the index of the line containing the $i$th cusp.

If $x_1 = 2$, then $x_2 = 1$ and the number of ways to complete the sequence is $a_{n-2}$. If $x_1 = 3$, then the number of ways to complete the sequence is $a_{n-1}$. So $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ and $a_0 = 1, a_1 = 2$.

## 7.1.17

### a

The number $a_n$ of sequences of numbers from $\{1, 3, 5\}$ that sum to $n$ satisfies the recurrence $a_n = a_{n-1} + a_{n-3} + a_{n-5}$ for $n \geq 5$ and $a_0 = a_1 = a_2 = 1, a_3 = 2, a_4 = 3$, since if such a sequence begins with $i$ then there are $a_{n-i}$ ways to complete the sequence.

### b

If we add the restriction that no 5 may be followed by a 1, then the recurrence becomes $a_n = a_{n-1} + a_{n-3} + a_{n-5} - a_{n-6}$, since we must subtract those cases where a 5 is followed by a 1. We also need to add one more initial condition: $a_5 = 5$.

## 7.1.35

To find the number $a_n$ of $n$-bit sequences with exactly 1 pair of consecutive 0s, first note that the number $b_n$ of $n$-bit sequences with no consecutive 0s satisfies the recurrence $b_n = b_{n-1} + b_{n-2}, b_0 = 1, b_1 = 2$, since if such a sequence begins with a 0 then the next bit must be a 1 and then there are $b_{n-2}$ ways to complete the sequence, and if such a sequence begins with a 1 then there are $b_{n-1}$ ways to complete the sequence.

Then $a_n = a_{n-1} + a_{n-2} + b_{n-3}, a_0 = a_1 = 0, a_2 = 1$ since if such a sequence begins with a 001 then there are $b_{n-3}$ ways to complete the sequence, if such a sequence begins with a 01 then there are $a_{n-2}$ ways to complete the sequence, and if such a sequence begins with 1 then there are $a_{n-1}$ ways to complete the sequence.

## 7.1.41

Let us build a vocabulary of simple properties of the $\Delta$ operator to help simplify later calculations:

| | |
|---|---|
| $\Delta(a_n + b_n) = \Delta a_n + \Delta b_n$ | linearity |
| $\Delta(ca_n) = c\Delta a_n$ | linearity |
| $\Delta(a_n b_n) = (\Delta a_n)b_n + a_{n-1}\Delta b_n$ | product rule |
| $\Delta(n) = 1$ | |
| $\Delta(c) = 0$ | |

For example,

$$\begin{aligned}\Delta(3n^2) &= 3\Delta(n^2) \\ &= 3(\Delta(n)n + (n-1)\Delta(n)) \\ &= 3(n+n-1) \\ &= 6n-3\end{aligned}$$

**a**

$$\begin{aligned}\Delta(3n+2) &= 3\Delta(n) = 3 \\ \Delta^2(3n+2) &= \Delta(3) = 0\end{aligned}$$

**b**

$$\begin{aligned}\Delta(n^2) &= \Delta(n)n + (n-1)\Delta(n) = 2n-1 \\ \Delta^2(n^2) &= \Delta(2n-1) = 2\end{aligned}$$

**c**

$$\begin{aligned}\Delta n^3 &= \Delta(n)n^2 + (n-1)\Delta(n^2) = n^2 + (n-1)(2n-1) = 3n^2 - 3n + 1 \\ \Delta^2(n^3) &= \Delta(3n^2 - 3n + 1) = 3\Delta(n^2) - 3\Delta(n) = 3(2n-1) - 3 = 6n - 6\end{aligned}$$

Note that the answer in the book for this problem makes no sense.

## 7.2.9

**a**

The following algorithm finds the largest value in a sequence of $n \geq 1$ values by using divide-and-conquer.

$\max(a[1, \ldots, n])$
  if $n = 1$, return $a[1]$
  $x \leftarrow \max(a, 1, \lfloor \frac{n}{2} \rfloor)$
  $y \leftarrow \max(a, \lfloor \frac{n}{2} \rfloor + 1, n)$
  if $x \geq y$, return $x$
  return $y$

The initial call is $\max(a, 1, n)$. Note that this algorithm is asymptotically as fast as linear search. In fact it is worse than linear search since it must pay the space overhead to maintain the call stack and pay the time to branch at each function call and return.

**b**

The number $c_n$ of comparisons that the above function uses satisfies the recurrence $c_n = c_{\lfloor \frac{n}{2} \rfloor} + c_{\lceil \frac{n}{2} \rceil} + 1, c_1 = 0$.

**c**

$c_n = n - 1$ solves the above. This was easily guessed since it is the number of internal nodes in a full binary tree with $n$ leaves.

## 7.3.3

**a**

The homogeneous equation $a_n = 3a_{n-1} + 4a_{n-2}$ has a corresponding characteristic polynomial equation $x^2 = 3x + 4$, which has roots $-1, 4$ from the quadratic formula. So any solution has the form $A(-1)^n + B4^n$. To satisfy the initial conditions $a_0 = a_1 = 1$, we must set $A = \frac{3}{5}, B = \frac{2}{5}$. So $a_n = \frac{3}{5}(-1)^n + \frac{2}{5}4^n$.

**b**

The recurrence $a_n = a_{n-2}, a_0 = a_1 = 1$ has solution $a_n = 1$, which can be easily guessed and then verified.

**c**

The characteristic polynomial associated with the homogeneous recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ has the double root 1. So $a_n$ is a linear polynomial. If $a_0 = a_1 = 2$, then $a_n = 2$.

**d**

Similarly, the characteristic polynomial for $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ has the triple root 1, and so $a_n$ is a quadratic polynomial. If $a_0 = a_1 = 1, a_2 = 2$, then $a_n = \frac{1}{2}n^2 - \frac{1}{2}n + 1$.

## 7.3.5

The number $a_n$ of ways of stacking flags on an $n$-foot tall flagpole using 2-foot tall red flags, 1-foot tall yellow flags, and 1-foot tall blue flags satisfies the recurrence $a_n = 2a_{n-1} + a_{n-2}, a_0 = 1, a_1 = 2$, the characteristic polynomial of which has roots $1 \pm \sqrt{2}$. So $a_n$ has the form $A(1 - \sqrt{2})^n + B(1 + \sqrt{2})^n$. By using the initial conditions, we find $A = \frac{1}{2} - \frac{1}{2\sqrt{2}}, B = \frac{1}{2} + \frac{1}{2\sqrt{2}}$, and so $a_n = (\frac{1}{2} - \frac{1}{2\sqrt{2}})(1 - \sqrt{2})^n + (\frac{1}{2} + \frac{1}{2\sqrt{2}})(1 + \sqrt{2})^n$.

We can drop the first term and round the second to the nearest integer since $a_n$ is an integer and the first term is always $< \frac{1}{2}$ in absolute value. So $a_n = \text{round}((\frac{1}{2} + \frac{1}{2\sqrt{2}})(1 + \sqrt{2})^n)$.

## 7.3.7

Suppose $\Delta p_n = 2\Delta p_{n-1}, p_0 = 1, p_1 = 4$. Then $\Delta p_1 = 3$ and so $\Delta p_i = 3 \cdot 2^{i-1}$. So $p_n = \sum_{i=1}^{n} \Delta p_i + p_0 = \sum_{i=1}^{n} 3 \cdot 2^{i-1} + 1 = 3 \cdot (2^n - 1) + 1$.

## 7.4.1

### a

If $a_n = a_{n-1} + 3(n-1), a_0 = 1$, then we guess that $a_n$ has the form $An^2 + Bn + C$ and solve for $A, B, C$. We get $A = \frac{3}{2}, B = -\frac{3}{2}, C = 1$. So $a_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$.

### b

If $a_n = a_{n-1} + n(n-1), a_0 = 3$, then we guess $a_n$ is cubic and solve for its coefficients to get $a_n = \frac{1}{3}n^3 - \frac{1}{3}n + 3$.

### c

If $a_n = a_{n-1} + 3n^2, a_0 = 10$, then we guess $a_n$ is cubic and solve for its coefficients to get $a_n = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n + 10$.

## 7.4.9

### a

If $a_n = 3a_{n-1} - 2, a_0 = 0$, then a particular solution to the inhomogeneous equation is 1 and the characteristic polynomial of the homogeneous equation has the single root 3. So a solution must be of the form $A3^n + 1$. By using the initial condition we find $A = -1$. So $a_n = -3^n + 1$.

### b

If $a_n = 2a_{n-1} + (-1)^n, a_0 = 2$, then we guess that a particular solution to the inhomogeneous equation has the form $B(-1)^n$ and solve for $B$, getting $B = \frac{1}{3}$. The characteristic polynomial of the homogeneous equation has the single root 2. So a solution must be of the form $A2^n + \frac{1}{3}(-1)^n$. By using the initial condition we find $A = \frac{5}{3}$. So $a_n = \frac{5}{3}2^n + \frac{1}{3}(-1)^n$.

Since $a_n$ is integer and the second term above is $< \frac{1}{2}$ in absolute value, we could express $a_n$ as round$(\frac{5}{3}2^n)$.

### c

If $a_n = 2a_{n-1} + n, a_0 = 1$, then we guess that a particular solution to the inhomogeneous equation is linear and solve for its coefficients to get $-n - 2$. The characteristic polynomial of the homogeneous equation has the single root 2, so a solution must be of the form $A2^n - n - 2$. By using the initial condition we find $A = 3$. So $a_n = 3 \cdot 2^n - n - 2$.

**d**

If $a_n = 2a_{n-1} + 2n^2, a_0 = 3$, then we guess that a particular solution to the inhomogeneous equation is quadratic and solve for its coefficients to get $-2n^2 - 8n - 12$. The characteristic polynomial for the homogeneous equation has the single root 2 and so $a_n$ has the form $A2^n - 2n^2 - 8n - 12$. By using the initial condition we find $A = 15$. So $a_n = 15 \cdot 2^n - 2n^2 - 8n - 12$.

## 7.4.13

Suppose $\Delta^2 a_n = 10 \cdot 2^n, a_0 = 20, a_1 = 1020$. Then

$$\begin{aligned}\Delta a_n &= \sum_{i=2}^{n} 10 \cdot 2^i + \Delta a_1 \\ &= 10(2^{n+1} - 4) + 1000 \\ &= 20 \cdot 2^n + 960\end{aligned}$$

and

$$\begin{aligned}a_n &= \sum_{i=1}^{n} \Delta a_i + a_0 \\ &= \sum_{i=1}^{n} (20 \cdot 2^i + 960) + 20 \\ &= 20 \cdot (2^{n+1} - 2) + 960n + 20 \\ &= 40 \cdot 2^n + 960n - 20.\end{aligned}$$