CSE 21 Homework 3 Solutions

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Exercises from Applied Combinatorics

8.1.13

a

Suppose that $T, P, B \subseteq S$ and

$$\begin{aligned}
&|S| = 200 \\
&|T| = |P| = |B| = 80 \\
&|T \cap P| = |T \cap B| = |P \cap B| = 30 \\
&|T \cap P \cap B| = 15.
\end{aligned}$$

Then

$$\begin{aligned}
|S - (T \cup P \cup B)| &= |S| - |T| - |P| - |B| \\
&\quad + |T \cap P| + |T \cap B| + |P \cap B| \\
&\quad - |T \cap P \cap B| \\
&= 200 - 3 \cdot 80 + 3 \cdot 30 - 15 \\
&= 35.
\end{aligned}$$

b

Also

$$\begin{aligned}
|P - (T \cup B)| &= |P| - |T \cap P| - |P \cap B| + |T \cap P \cap B| \\
&= 80 - 30 - 30 + 15 \\
&= 35.
\end{aligned}$$

8.1.17

The number N of ways to assign 20 different people to 3 different rooms with at least 1 person in each room is $3^{20} - \binom{3}{1}2^{20} + \binom{3}{2}1^{20}$. To see this, let A_C be the number of ways to assign people to rooms where each room in the collection C has 0 people. Then $N = A - A_1 - A_2 - A_3 + A_{1,2} + A_{1,3} + A_{2,3} - A_{1,2,3}$. Of course $A_{1,2,3} = 0$.

8.1.19

To count the number A of arrangements of the letters of MURMUR in which no 2 consecutive letters are the same, let S_l be the number of arrangements of the letters of MURMUR in which at least the l's are together. (If l is a collection of letters, then S_l is the number of arrangements in which each letter x in the collection is next to another x.) Then

$$\begin{aligned} A &= |S| - |S_M| - |S_U| - |S_R| \\ &\quad + |S_{M,U}| + |S_{M,R}| + |S_{U,R}| \\ &\quad - |S_{M,U,R}| \\ &= \frac{6!}{2!^3} - 3 \cdot \frac{5!}{2!^2} + 3 \cdot \frac{4!}{2!} - 3! \\ &= 30. \end{aligned}$$

8.1.25

To count the number A of arrangements of the letters of TAMELY in which either the T is before the A or the A is before the M or the M is before the E, number these 3 constraints as $1, 2, 3$ and let S_C be the number of arrangements of the letters in which at least the constraints in the collection C are met. Then

$$\begin{aligned} A &= S_1 + S_2 + S_3 - S_{1,2} - S_{1,3} - S_{2,3} + S_{1,2,3} \\ &= 3 \cdot \binom{6}{2} 4! - \binom{6}{3} 3! - \binom{6}{2,2,2} 2! - \binom{6}{3} 3! + \binom{6}{4} 2! \\ &= 690. \end{aligned}$$

8.1.33

Suppose that $S, W, C \subseteq P$ and that

$$\begin{aligned} &|P| = 150 \\ &|S \cup W| = 90 \\ &|W \cup C| = 80 \\ &|S \cup C| = 100 \\ &|C \cap (S \cup W)| = 50 \\ &|(S \cap W) \cup (S \cap C) \cup (W \cap C)| = 60 \\ &|S \cap W \cap C| = 20. \end{aligned}$$

We partition P as follows:

$$\begin{aligned}
N_0 &= |P - (S \cup W \cup C)| \\
N_1 &= |S - (W \cup C)| \\
N_2 &= |W - (S \cup C)| \\
N_3 &= |C - (S \cup W)| \\
N_4 &= |(S \cap W) - C| \\
N_5 &= |(S \cap C) - W| \\
N_6 &= |(W \cap C) - S| \\
N_7 &= |S \cap W \cap C|.
\end{aligned}$$

Then the constraints imply

$$\begin{array}{rrrrr}
N_0+N_1+ & N_2+N_3+ & N_4+N_5+ & N_6+N_7 & =150 \\
N_1+ & N_2+ & N_4+N_5+ & N_6+N_7 & =90 \\
 & N_2+N_3+ & N_4+N_5+ & N_6+N_7 & =80 \\
N_1 & +N_3+ & N_4+N_5+ & N_6+N_7 & =100 \\
 & & N_5+ & N_6+N_7 & =50 \\
 & & N_4+N_5+ & N_6+N_7 & =60 \\
 & & & N_7 & =20.
\end{array}$$

Although the system is underconstrained (more variables than equations) we can still compute most of the values of the N_i by Gaussian elimination:

$$\begin{aligned}
N_7 &= 20 \\
N_5 + N_6 &= 30 \\
N_4 &= 10 \\
N_3 &= 15 \\
N_2 &= 5 \\
N_1 &= 25 \\
N_0 &= 45.
\end{aligned}$$

So

$$\begin{aligned}
&|(S - (W \cup C)) \cup (W - (S \cup C)) \cup (C - (S \cup W))| \\
&\qquad = N_1 + N_2 + N_3 = 25 + 5 + 15 = 45.
\end{aligned}$$

The book asks you to compute $|W| = N_2 + N_4 + N_6 + N_7 = 5 + 10 + N_6 + 20 = N_6 + 35$, but this is not possible from the data given: it is consistent with the data for $|W|$ to be any integer between 35 and 65. Also, the answer at the back of the book makes no sense.

Exercises from Bender

CL4.2

If $E_1, \ldots, E_{38}$ are disjoint events and $\forall i \in \{1, \ldots, 37\}\ Pr(E_i) = p$ and $Pr(E_{38}) = 1.05p$, then $37p + 1.05p = 1 \Rightarrow p = \frac{1}{38.05}$ and so $1.05p = \frac{1.05}{38.05} \approx .0276$.

CL4.6

For each $k \in \mathbb{N}$, let $[k]$ be the set of integers $\{1, \ldots, k\}$. Also if A is a set, let A^k be the cartesian product of k factors of A.

a

One sample space that models when a fair die is cast k times is $S = [6]^k$.

The probability that the sum of the values is even is $\frac{1}{2}$ since this is the probability that the last value is congruent to the sum of the first $k-1$ values modulo 2.

b

One sample space that models when a card is drawn from a deck n times with replacement is $(\{2, \ldots, 10, J, Q, K, A\} \times \{S, C, D, H\})^n$.

The probability that a K does not appear on any of the n draws is $(1 - \frac{4}{52})^n$, since K must fail to appear n times independently.

The probability that at least 1 K appears in n draws is $1 - (1 - \frac{4}{52})^n$.

c

We can model drawing 2 marbles without replacement from an urn containing 3 white, 4 red, 5 blue marbles with the sample space $S = \{(x, y) \mid x, y \in [3] \times \{W\} \cup [4] \times \{R\} \cup [5] \times \{B\}, x \neq y\}$.

To compute the probability p that both marbles are red, let X_i be the color of the ith marble drawn, $i = 1, 2$. Then $p = Pr(X_1 = X_2 = R) = Pr(X_2 = R \mid X_1 = R) Pr(X_1 = R) = \frac{3}{11} \cdot \frac{4}{12} = \frac{1}{11}$.

CL4.7

Let X be the number of defective light bulbs of those chosen when 6 bulbs are chosen from a lot of 15, of which 5 are defective. Then X is hypergeometric and $Pr(X = 3) = \frac{\binom{5}{3}\binom{10}{3}}{\binom{15}{6}} \approx .24$, since in order for X to be 3 we must choose 3 of the 5 defectives and 3 of the 10 nondefectives.

CL4.14

To compute the probability p that when 3 boys and 3 girls are aligned at random that the 3 girls are together, note that the number of strings of the form $b^{x_1}g^3b^{x_2}$ in which $x_1 + x_2 = 3, x_i \geq 0$ is 4. So $p = \frac{4}{\binom{6}{3}} = \frac{1}{5}$.