CSE 21 Homework 2 Solutions

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Exercises from Applied Combinatorics

5.3.7

Suppose we want to choose a committee of D indistinguishable democrats, R indistinguishable republicans, I indistinguishable independents such that $2 \leq D, R, I$ and $D + R + I = 15$. This is the same as the number of nonnegative (d, r, i) triples of integers whose sum is $15-6$. So the number of such committees is $\binom{15-6+3-1}{3-1} = 55$.

Suppose that we also require $D, R, I \leq 7$. This is the same as the number A of (d, r, i) triples of integers between 0 and 5 whose sum is $15 - 6$. Given that the sum and lower bound constraints are satisfied, it is only possible to violate at most 1 of the upper bound constraints, of which there are $\binom{3}{1}$. To violate, say, $d \leq 5$ is the same as forcing $d \geq 6$. So the number of ways to satisfy all of the constraints except for 1 specific upper bound constraint is $\binom{15-6-6+3-1}{3-1}$. So $A = \binom{15-6+3-1}{3-1} - \binom{3}{1}\binom{15-6-6+3-1}{3-1} = 25$.

5.3.17

The number of arrangements of the letters of REPETITION in which the first E occurs before the first T is half of them, i.e. $\frac{1}{2} \cdot \frac{10!}{2!^3} = 226800$. To see that half of the arrangements have the first E before the first T, note that we can bijectively map such arrangements to the arrangements in which the first T occurs before the first E by interchanging all E's with T's.

5.3.21

a

The number of arrangements of the letters of MISSISSIPPI in which the M is followed immediately by an I is $\frac{10!}{4!3!2!} = 12600$, since we can simply consider the word to contain the superletter 'MI'. Then there are 4 S's, 3 I's, and 2 P's.

b

To count the number of arrangements of the letters of MISSISSIPPI in which the M is next to an I, we count the number of ways in which the M is followed by an I, add the ways in which the M is preceded by an I, and subtract the number of ways in which the M is both followed and preceded by an I, i.e. $2 \cdot \frac{10!}{4!3!2!} - \frac{9!}{4!2!2!} = \frac{9!}{4!2!2!}(\frac{2 \cdot 10}{3} - 1) = 21420$.

5.3.29

a

When n coins are flipped the probability that the 1st head occurs after exactly m tails is

$$\begin{cases} (\frac{1}{2})^{m+1} & \text{if } m+1 \leq n \\ 0 & \text{else} \end{cases},$$

since the ith flip is exactly what it must be for this event to occur with probability $\frac{1}{2}$, or 0 if there is no ith flip.

b

When n coins are flipped the probability that the ith head comes after exactly m tails is

$$\begin{cases} \binom{m+i-1}{i-1}(\frac{1}{2})^{m+i} & \text{if } m+i \leq n \\ 0 & \text{else} \end{cases},$$

since, assuming $m + i \leq n$, any particular sequence of $m + i$ flips occurs with probability $(\frac{1}{2})^{m+i}$ and the number of sequences of flips of length $m + i$ with $i - 1$ heads in positions $1, \ldots, m + i - 1$, a head in position $m + i$, and tails in the other positions is $\binom{m+i-1}{i-1}$.

5.3.31

This problem can be solved in many ways, with some ways more acceptable than others. In general, an answer A is more acceptable than B if the expression A has fewer operators. [1] In particular, an answer with doubly or triply nested summations over large indexes of summation - or worse yet, when the inner indexes of summation depend on the outer indexes of summation - is less acceptable than a single sum with a small number of terms. Large complex expressions generally imply a lack of understanding of a problem. E.g. in $\sum_{j=0}^{n} \sum_{i=0}^{j} \binom{j}{i} 2^{i+j} = \frac{6^{n+1}-1}{6-1}$, the left-hand side is less acceptable than the right.

[1] important exception: we often want to see a formula written like $\binom{6}{3} - \binom{4}{2}$ because if we replaced the constants of the problem by variables, such formulas often remain valid.

To count the number of arrangements of 7 as, 8 bs, 3cs, 6ds with no occurrence of ca nor cc, we note that such an arrangement must either end in a c or not. If an arrangement does not end in a c, then every occurrence of a c must be followed by a b or a d. We further divide into cases based on the number k of cs followed by b.

The number of arrangements that do not end in c, do not contain ca nor cc, and have k cs followed by b is $\binom{21}{k,3-k,7,8-k,6-(3-k)}$. The sequence $k, 3-k, 7, 8-k, 6-(3-k)$ represents the number of cbs, cds, as, bs, ds.

The number of arrangements that do end in a c, do not contain ca nor cc, and have k cs followed by b is $\binom{21}{k,2-k,7,8-k,6-(2-k)}$. The sequence $k, 2-k, 7, 8-k, 6-(2-k)$ represents the number of cbs, cds, as, bs, ds.

So the number of arrangements that do not contain ca nor cc is

$$\sum_{k=0}^{3} \binom{21}{k, 3-k, 7, 8-k, 6-(3-k)} + \sum_{k=0}^{2} \binom{21}{k, 2-k, 7, 8-k, 6-(2-k)}.$$

Note that this expression involves only 7 terms. (I think the book has an error here: it claims the 2nd set of terms should have a 20 in the top part of the multinomial coefficient.)

5.4.3

a

The probability that a bridge deal has 4 spades, 3 hearts, 3 diamonds, 3 clubs is $\frac{\binom{13}{4}\binom{13}{3}^3}{\binom{52}{13}} \approx .026$, since such a hand can be formed by choosing 4 of the 13 spades, 3 of the 13 hearts, etc.

b

The probability that in a bridge deal North and South each have 5 spades, West has 2 spades, East has 1 spade is $\frac{\binom{13}{5,5,2,1}\binom{52-13}{8,8,11,12}}{\binom{52}{13,13,13,13}}$, since this is the quotient of the product of the number of ways of so distributing the spades and the number of ways of the remaining $52-13$ cards and the number of ways of distributing all 52 cards.

c

The following answer is tempting, but wrong: "The probability that there is a player who gets all of the aces is $4^{-3} = \frac{1}{64}$, since this is the probability that after the 1st ace is distributed, the remaining 3 go to the same player." The problem is that the aces are *not* distributed independently: if a player gets an ace, he is *less* likely to get another.

The probability that a player gets all 4 aces is actually $\frac{4 \cdot \binom{52-4}{13-4}}{\binom{52}{13}} = \frac{44}{4165}$, since there are 4 players who could get all 4 aces and the probability that a specific

player gets all 4 aces is the number of ways to choose the remaining $13-4$ cards divided by the number of ways of choosing 13 cards.

d

To count the number of deals in which each player has a $(4,3,3,3)$ division of suits, we note that in such a deal, each player corresponds uniquely to the suit that he has a plurality (i.e. 4) of. So the number A of such deals is 4! times the number B of such deals with a fixed assignment of plurality suits to players. $B = \binom{13}{4,3,3,3}^4$ since for each of the 4 suits, we must distribute all 13 of its cards so that the player designated to receive 4 gets 4 and the remaining players each get 3. So

$$A = \frac{4!\binom{13}{4,3,3,3}^4}{\binom{52}{13,13,13,13}} \approx .0009.$$

I think the book is in error here.

5.4.9

In a sequence of 20 coin flips, the probability that we get no consecutive tails *given* that we get 14 heads, 6 tails is the number of ways of inserting 6 tails into $14+1$ locations without repetition divided by the number of sequences of 14 heads and 6 tails, i.e. $\frac{\binom{14+1}{6}}{\binom{14+6}{6}} \approx .129$.

5.4.11

The number of integer solutions to $x_1 + x_2 + x_3 = 0, x_i \geq -5$ is $\binom{18-1}{3-1} = 136$, since this is the same as the number of integer solutions to $y_1+y_2+y_3 = 18, y_i \geq 1$ by setting $y_i = x_i + 6$.

5.4.27

a

The number of arrangements of 8 letters chosen from piles of as, bs, cs is the number of distributions of 8 positions into buckets labeled a, b, c.

b

The number of arrangements of 2 as, 3 bs, 4 cs is the number of distributions of 2 separators among (2+3+4)+1 positions.

c

The number of arrangements of 10 letters chosen from piles of as, bs, cs, ds with the same number of as as bs is the number of distributions of 10 locations into buckets labeled a, b, c, d such that the number of locations put into bucket a is the same as the number put into bucket b.

d

The number of arrangements of 4 letters chosen from 2 as, 3 bs, 4 cs is the number of distributions of 4 positions into buckets labeled a, b, c such that the number of positions in bucket a is ≤ 2 and the number of positions in bucket b is ≤ 3.

5.4.55

The number of 6-subsets of $\{1, \ldots, 20\}$ without consecutive integers is the same as the number A of 7-tuples $(g_1, \ldots, g_7)$ of integers such that $g_1 + \cdots + g_7 + 6 = 20$ and $g_1, g_7 \geq 0, g_2, \ldots, g_6 \geq 1$. The idea here is that g_i represents the size of the ith "gap" in the sequence; so $(g_1, \ldots, g_7)$ corresponds to the subset not containing the first g_1 integers, then containing $g_1 + 1$, then not containing the next g_2 integers, etc. A is the same as the number of 7-tuples of positive integers whose sum is $20 - 6 + 2$ (by adding 1 to each of g_1, g_7) and this is $\binom{20-6+2-1}{7-1} = 5005$.

5.4.57

The number A of arrangements of n 0s, m 1s with k runs of 0s is the number of integer solutions to the system

$$\sum_{i=1}^{k} r_i = n, \quad r_i \geq 1$$
$$\sum_{i=0}^{k} s_i = m, \quad s_0, s_k \geq 0, \forall i \notin \{0, k\}\ s_i \geq 1,$$

which can be seen by letting r_i be the size of the ith run of 0s and s_i be the size of the ith run of 1s. So $A = \binom{n-1}{k-1}\binom{m+1}{k}$.

5.4.59

The number A of binary sequences of length n that contain exactly m occurrences of the pattern 01 is the same as the number of integer solutions to the

system

$$\sum_{i=0}^{m} x_i + \sum_{i=0}^{m} y_i = n$$
$$\forall 1 \leq i \leq m \; x_i, y_i \geq 1$$
$$x_0, y_0 \geq 0,$$

which can be seen by noting that a binary sequence contains exactly m occurrences of 01 iff it has the form $1^{y_0}0^{x_1}1^{y_1}0^{x_2}1^{y_2}\cdots 0^{x_m}1^{y_m}0^{x_0}$ with the above restrictions on the values of the exponents. So $A = \binom{n+2-1}{2m+2-1}$.

Exercises from Bender

3.5

The number of arrangements of a deck of cards if the suits are ignored is $\binom{52}{13,13,13,13} \approx 5 \cdot 10^{28}$, since this is the number of ways of distributing the 52 locations into 4 distinct piles, each of size 13.

3.6

c

To count the number of sequences of letters

1. using only A,I,L,S,T
2. 7 letters long
3. beginning and ending with consonants
4. with no adjacent vowels
5. without 3 adjacent consonants
6. with no single consonant twice in a row,

we use ad hoc methods. (also note that the book is in error when is claims that LASLASS is such a sequence, since it violates rule 6.) First, note that the number of vowels in such a sequence must be either 2 or 3: any less would force violation of rule 5, any more would force violation of rule 4.

If there are 2 vowels, then they must be in one of the following 3 arrangements: CVCCVCC, CCVCVCC, CCVCCVC. In each such case, there are exactly $3^3 2^4$ sequences because each run of consonants has 1 3-way choice and the rest are 2-way choices and the vowels are all 2-way choices. So the number of sequences with 2 vowels is $3^4 2^4$.

If there are 3 vowels, then they must be in the arrangement CVCVCVC and there are $3^4 2^3$ such sequences.

The total is $3^4 2^4 + 3^4 2^3 = 1944$.

a

The first 4 sequences in lexicographical order are LALALAL, LALALAS, LALALAT, LALALIT.

b

The last 4 sequences in lexicographical order are TSITSAT, TSITSIL, TSITSIS, TSITSIT.

3.10

a

There is only one permutation $a_1, \ldots, a_9$ of $1, \ldots, 9$ such that $\forall 1 \leq i < 9\ a_i < a_{i+1}$, namely, the identity permutation.

b

There are $\binom{9}{5} - 1$ permutations $a_1, \ldots, a_9$ of $1, \ldots, 9$ such that $\forall i \in [9] - \{5\}\ a_i < a_{i+1}$ and $a_5 > a_6$. (If you interpret the question as asking for how many permutations satisfy $\forall i \in [9] - \{5\}\ a_i < a_{i+1}$, without requiring that $a_5 > a_6$, then your answer will be larger by 1.) To see this, note that any such permutation can be constructed by choosing 5 of the 9 numbers to form the set $\{a_1, \ldots, a_5\}$, and then the remaining numbers form the set $\{a_6, \ldots, a_9\}$. Since $a_1 < \cdots < a_5$ and $a_6 < \cdots < a_9$, the permutation is completely determined. We subtract 1 to discount the identity permutation.