CSE 21 Homework 1 Solutions

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Exercises from Applied Combinatorics

5.1.1

a

The number of ways to pick a sequence of 2 different letters of the alphabet that appear in the word BOAT is $P(4,2) = 12$. The number of ways to pick a sequence of 2 different letters of the alphabet that appear in the word MATHEMATICS is $P(8,2) = 56$, since MATHEMATICS has only 8 distinct letters.

b

The number of pairs (x, y) where x is a vowel and y is a consonant from the letters of BOAT is $2 \cdot 2 = 4$. The number of pairs (x, y) where x is a vowel and y is a consonant from the letters of MATHEMATICS is $3 \cdot 5 = 15$.

5.1.9

a

The number of ways to pick 2 different cards from a 52 card deck such that the 1st card is an ace and the 2nd card is not a queen is $4 \cdot 47 = 188$, since there are 4 aces and 47 non-queens remaining once an ace is removed from the deck.

b

The number of ways to pick 2 different cards from a 52 card deck such that the 1st card is a spade and the 2nd card is not a queen is $12 \cdot 47 + 1 \cdot 48 = 612$, since there are 12 spade non-queens and 47 non-queens left if one is removed and 1 spade queen and 48 non-queens left if the spade queen is removed.

5.1.11

The number of ways of rolling 2 (differently colored) dice to yield a sum divisible by 3 is $2 + 5 + 4 + 1 = 12$, since the sum is divisible by 3 iff the sum is

one of $3, 6, 9, 12$, and the number of ways of getting each of these is $2, 5, 4, 1$, respectively.

5.1.15

The probability that the top 2 cards in a (uniformly randomly) shuffled deck do not form a pair is $\frac{51-3}{51} = \frac{16}{17}$, since after the 1st card is selected, $51 - 3$ out of the 51 remaining cards do not match its rank.

5.1.19

The number of 3-letter sequences without repetition of the letters a, b, c, d, e, f that have an e is $P(6,3) - P(5,3) = 60$, since there are $P(6,3)$ 3-letter sequences without repetition and $P(5,3)$ of those avoid using e.

5.1.25

The number of sequences of length 5 using only the symbols a, b and in which both symbols appear is $2^5 - 2$. Given such a sequence, if we replace a, b by 2 elements, say a_0, b_0, from $0, \ldots, 9$ where $a_0 < b_0$, then we will obtain a sequence of length 5 of digits from $0, \ldots, 9$ in which exactly 2 digits appear. Furthermore, every such sequence can be generated in this way and every 2 distinct choices yields a distinct sequence.

The latter is not so obvious. To see this, let x, x' be 2 sequences of length 5 using a, b and $a_0 < b_0, a_0' < b_0'$ be digits from $0, \ldots, 9$ and suppose $(x, a_0, b_0) \neq (x', a_0', b_0')$. Now suppose indirectly that these choices generate the same sequence α. Clearly we cannot have $(a_0, b_0) \neq (a_0', b_0')$ since then α contains ≥ 3 distinct digits. So we must have $x \neq x'$. Let i be some position in which x, x' differ. Wlog $x_i = a, x_i' = b$. But these choices generate the same sequence, so we must have $a_0 = b_0'$. But then we have the contradiction $b_0 > a_0 = b_0' > a_0'$.

So the number of sequences of digits from $0, \ldots, 9$ in which exactly 2 digits appear is $(2^5 - 2) \cdot C(10, 2) = 1350$.

5.1.35

The number A of pairs (x, y) of distinct numbers in $\{1, \ldots, 50\}$ such that one is twice the other is twice the number of pairs (x, y) in $\{1, \ldots, 50\}$ where $y = 2x$. So $A = 2 \cdot 25$. The number of pairs of distinct numbers in $\{1, \ldots, 50\}$ is $50 \cdot 49$.

So the probability that 2 numbers selected randomly without replacement from $1, \ldots, 50$ have the property that one is twice the other is $\frac{2 \cdot 25}{50 \cdot 49} = \frac{1}{49}$.

5.1.45

The number of nonempty subsets of a set of size 10 is $2^{10} - 1 = 1023$, since each element is either in the subset or not and we subtract 1 since we don't count the empty set.

5.2.3

The number of ways to distribute 9 distinct books among 15 children such that no child gets more than 1 book is $P(15,9) = 1816214400$, since the 1st book goes to 1 of 15 children, the 2nd book goes to 1 of 14 children, and in general the ith book goes to 1 of $15 - i + 1$ children for $1 \leq i \leq 9$.

5.2.7

The number of 5-person basketball teams that can be formed from 10 players is $C(10,5) = 252$. Supposing that there is just 1 weakest and 1 strongest player, the number of such teams containing both is $C(8,3) = 56$ since we must pick 3 players of the 8 remaining after the weakest and strongest are already chosen.

5.2.11

a

The number of ways of distributing 10 objects to 5 people so that each person gets exactly 2 objects is $\frac{10!}{2!^5} = 113400$ since we can create such a distribution by first arranging the 10 objects as $x_1, \ldots, x_{10}$, giving x_{2i-1}, x_{2i} to person i, and then ignoring the order of the 2 objects given to each person.

b

The number of ways of distributing 5 distinct left-handed gloves and 5 distinct right-handed gloves to 5 people such that each gets a left-handed glove and a right-handed glove is $5!^2 = 14400$, since we can distribute the gloves first by arranging the left-handed gloves into a sequence of 5 and then arranging the right-handed gloves into a sequence of 5.

5.2.17

a

To count the number of committees that can be formed from 4 men and 6 women with ≥ 2 men and at least twice as many women as men, let M, W be the number of men, women chosen. Note that we have the restrictions

$$2 \leq M \leq 4$$
$$2M \leq W \leq 6.$$

and so the legal (M, W) pairs are $(2,4), (2,5), (2,6), (3,6)$. The number of committees for each of these 4 cases is $\binom{4}{2}\binom{6}{4}, \binom{4}{2}\binom{6}{5}, \binom{4}{2}\binom{6}{6}, \binom{4}{3}\binom{6}{6}$, respectively, the total of which is 136.

b

The number of committees of size between 3 and 5 that can be formed from 4 men and 6 women when 1 woman is excluded is $\binom{9}{3} + \binom{9}{4} + \binom{9}{5} = 336$, since we must choose $3, 4$ or 5 people from the remaining 9 candidates.

c

The number of committees of size 5 that can be formed from 10 people but not containing a specific group of 3 people is $\binom{10}{5} - \binom{7}{2} = 231$ since there are $\binom{10}{5}$ subsets of size 5 and $\binom{7}{2}$ subsets of size 5 containing a specific subset of size 3.

d

To count the number of committees of size 4 that can be formed from 4 men and 6 women where ≥ 2 women and not containing a specific man,woman pair, note that the number of men,women chosen must be one of $(2,2), (1,3), (0,4)$. The number of committees for each case without the restriction that the specific man,woman pair is excluded is $\binom{4}{2}\binom{6}{2}, \binom{4}{1}\binom{6}{3}, \binom{4}{0}\binom{6}{4}$. To form a committee of size 4 with at least 2 women and containing the specific man,woman pair, we must select the specific man,woman pair and then the number of extra men,women must be one of $(1,1), (0,2)$. The number of committees for each case is $\binom{3}{1}\binom{5}{1}, \binom{3}{0}\binom{5}{2}$.

So the number of committees of size 4 with at least 2 women and excluding the specific man,woman pair is $\binom{4}{2}\binom{6}{2} + \binom{4}{1}\binom{6}{3} + \binom{4}{0}\binom{6}{4} - (\binom{3}{1}\binom{5}{1} + \binom{3}{0}\binom{5}{2}) = 160$.

5.2.27

The smallest n such that there are at least 365 subsets of $\{1, \ldots, n\}$ of size 4 is $\min\{n \geq 0 \mid \binom{n}{4} \geq 365\} = 12$, which was easily be found by binary search since $\binom{n}{4}$ is an increasing function of n.

5.2.41

a

To choose a 5-card hand so that at least 1 of each of the 4 face values appears, we may choose one of the 4 face values to appear twice and the other 3 to appear once each, or we may choose each of the 4 face values to appear once and a single non-face value to appear. So the number of such hands is $4 \cdot \binom{4}{2}\binom{4}{1}^3 + \binom{4}{1}^4\binom{52-4\cdot 4}{1}$. The probability of choosing such a hand is then

$$\frac{4 \cdot \binom{4}{2}\binom{4}{1}^3 + \binom{4}{1}^4\binom{52-4\cdot 4}{1}}{\binom{52}{5}} = \frac{32}{7735} \approx .004.$$

b

To calculate the probability that a randomly selected 5-card hand has the same number of hearts as spades, note that the number of hearts,spades,others chosen must be one of $(0,0,5),(1,1,3),(2,2,1)$. The number of such cases is $\binom{13}{0}^2\binom{26}{5}$, $\binom{13}{1}^2\binom{26}{3}$, $\binom{13}{2}^2\binom{26}{1}$, respectively. So the desired probability is

$$\frac{\binom{13}{0}^2\binom{26}{5}+\binom{13}{1}^2\binom{26}{3}+\binom{13}{2}^2\binom{26}{1}}{\binom{52}{5}}=\frac{12757}{49980}\approx .26.$$

1 Exercises from Bender

1.5

The number of sequences of letters from $a,\ldots,z$ of length 4 containing at least one of a,e,i,o,u is the number of sequences of letters from $a,\ldots,z$ of length 4 minus the number of such sequences that fail to contain any of a,e,i,o,u; i.e. $26^4-(26-5)^4=262495$.

2.1

a

The number of ways to arrange 3 (distinct) boys and 4 (distinct) girls is $7!=5040$.

b

The number of ways to arrange 3 boys and 4 girls such that the boys are consecutive and the girls are consecutive is $2\cdot 3!\cdot 4!=288$, since either the boys come first or the girls, and then there are 3! ways of arranging the boys among themselves and 4! ways of arranging the girls among themselves.

c

The number of ways to arrange 3 boys and 4 girls such that the gender alternates is $3!\cdot 4!=144$ since these sequences are in one-to-one correspondence with the permutations of 4 girls and the permutations of 3 boys. The correspondence is to look at the girls in positions $1,3,5,7$ and then the boys in positions $2,4,6$.

2.2

a

The number of ways to arrange 3 (distinct) boys and 3 (distinct) girls is $6!=720$.

b

The number of ways to arrange 3 boys and 3 girls such that the boys are consecutive and the girls are consecutive is $2 \cdot 3! \cdot 3! = 72$, since either the boys come first or the girls, and then there are 3! ways of arranging the boys among themselves and 3! ways of arranging the girls among themselves.

c

If the girls and boys must alternate, then either the girls occupy positions $1, 3, 5$ or they occupy positions $2, 4, 6$. So the number of such arrangements is $2 \cdot 3! \cdot 3! = 72$.

2.7

a

The sequences of letters from R,L,F,I,E of length ≤ 6 that contain ≥ 1 L, begin with R, end with F, contain exactly one R and exactly one F must either have $1, 2, 3$ or 4 letters between the R and the F. The number of such sequences with i letters between the R and the F is $3^i - 2^i$ since there are 3^i sequences of letters from L,F,I of length i and 2^i of them avoid using the L. So the number of sequences of letters from R,L,F,I,E of length ≤ 6 that contain ≥ 1 L, begin with R, end with F, contain exactly one R and exactly one F is $3^1 - 2^1 + 3^2 - 2^2 + 3^3 - 2^3 + 3^4 - 2^4 = 90$.

b

The 5 sequences preceding RELIEF in dictionary order (going backwards) are

1. RELELF
2. RELEIF
3. RELEEF
4. REILLF
5. REILIF.