

CSE 21 Homework 1 Solutions

Chris Calabro

June 30, 2005

Exercises from Applied Combinatorics

5.1.1

a

The number of ways to pick a sequence of 2 different letters of the alphabet that appear in the word BOAT is $P(4, 2) = 12$. The number of ways to pick a sequence of 2 different letters of the alphabet that appear in the word MATHEMATICS is $P(8, 2) = 56$, since MATHEMATICS has only 8 distinct letters.

b

The number of pairs (x, y) where x is a vowel and y is a consonant from the letters of BOAT is $2 \cdot 2 = 4$. The number of pairs (x, y) where x is a vowel and y is a consonant from the letters of MATHEMATICS is $3 \cdot 5 = 15$.

5.1.9

a

The number of ways to pick 2 different cards from a 52 card deck such that the 1st card is an ace and the 2nd card is not a queen is $4 \cdot 47 = 188$, since there are 4 aces and 47 non-queens remaining once an ace is removed from the deck.

b

The number of ways to pick 2 different cards from a 52 card deck such that the 1st card is a spade and the 2nd card is not a queen is $12 \cdot 47 + 1 \cdot 48 = 612$, since there are 12 spade non-queens and 47 non-queens left if one is removed and 1 spade queen and 48 non-queens left if the spade queen is removed.

5.1.11

The number of ways of rolling 2 (differently colored) dice to yield a sum divisible by 3 is $2 + 5 + 4 + 1 = 12$, since the sum is divisible by 3 iff the sum is

one of 3, 6, 9, 12, and the number of ways of getting each of these is 2, 5, 4, 1, respectively.

5.1.15

The probability that the top 2 cards in a (uniformly randomly) shuffled deck do not form a pair is $\frac{51-3}{51} = \frac{16}{17}$, since after the 1st card is selected, 51 - 3 out of the 51 remaining cards do not match its rank.

5.1.19

The number of 3-letter sequences without repetition of the letters a, b, c, d, e, f that have an e is $P(6, 3) - P(5, 3) = 60$, since there are $P(6, 3)$ 3-letter sequences without repetition and $P(5, 3)$ of those avoid using e .

5.1.25

The number of sequences of length 5 using only the symbols a, b and in which both symbols appear is $2^5 - 2$. Given such a sequence, if we replace a, b by 2 elements, say a_0, b_0 , from $0, \dots, 9$ where $a_0 < b_0$, then we will obtain a sequence of length 5 of digits from $0, \dots, 9$ in which exactly 2 digits appear. Furthermore, every such sequence can be generated in this way and every 2 distinct choices yields a distinct sequence.

The latter is not so obvious. To see this, let x, x' be 2 sequences of length 5 using a, b and $a_0 < b_0, a'_0 < b'_0$ be digits from $0, \dots, 9$ and suppose $(x, a_0, b_0) \neq (x', a'_0, b'_0)$. Now suppose indirectly that these choices generate the same sequence α . Clearly we cannot have $(a_0, b_0) \neq (a'_0, b'_0)$ since then α contains ≥ 3 distinct digits. So we must have $x \neq x'$. Let i be some position in which x, x' differ. Wlog $x_i = a, x'_i = b$. But these choices generate the same sequence, so we must have $a_0 = b'_0$. But then we have the contradiction $b_0 > a_0 = b'_0 > a'_0$.

So the number of sequences of digits from $0, \dots, 9$ in which exactly 2 digits appear is $(2^5 - 2) \cdot C(10, 2) = 1350$.

5.1.35

The number A of pairs (x, y) of distinct numbers in $\{1, \dots, 50\}$ such that one is twice the other is twice the number of pairs (x, y) in $\{1, \dots, 50\}$ where $y = 2x$. So $A = 2 \cdot 25$. The number of pairs of distinct numbers in $\{1, \dots, 50\}$ is $50 \cdot 49$.

So the probability that 2 numbers selected randomly without replacement from $1, \dots, 50$ have the property that one is twice the other is $\frac{2 \cdot 25}{50 \cdot 49} = \frac{1}{49}$.

5.1.45

The number of nonempty subsets of a set of size 10 is $2^{10} - 1 = 1023$, since each element is either in the subset or not and we subtract 1 since we don't count the empty set.

5.2.3

The number of ways to distribute 9 distinct books among 15 children such that no child gets more than 1 book is $P(15, 9) = 1816214400$, since the 1st book goes to 1 of 15 children, the 2nd book goes to 1 of 14 children, and in general the i th book goes to 1 of $15 - i + 1$ children for $1 \leq i \leq 9$.

5.2.7

The number of 5-person basketball teams that can be formed from 10 players is $C(10, 5) = 252$. Supposing that there is just 1 weakest and 1 strongest player, the number of such teams containing both is $C(8, 3) = 56$ since we must pick 3 players of the 8 remaining after the weakest and strongest are already chosen.

5.2.11

a

The number of ways of distributing 10 objects to 5 people so that each person gets exactly 2 objects is $\frac{10!}{2!^5} = 113400$ since we can create such a distribution by first arranging the 10 objects as $x_1, \dots, x_{10}$, giving x_{2i-1}, x_{2i} to person i , and then ignoring the order of the 2 objects given to each person.

b

The number of ways of distributing 5 distinct left-handed gloves and 5 distinct right-handed gloves to 5 people such that each gets a left-handed glove and a right-handed glove is $5!^2 = 14400$, since we can distribute the gloves first by arranging the left-handed gloves into a sequence of 5 and then arranging the right-handed gloves into a sequence of 5.

5.2.17

a

To count the number of committees that can be formed from 4 men and 6 women with ≥ 2 men and at least twice as many women as men, let M, W be the number of men, women chosen. Note that we have the restrictions

$$\begin{aligned} 2 &\leq M \leq 4 \\ 2M &\leq W \leq 6. \end{aligned}$$

and so the legal (M, W) pairs are $(2, 4), (2, 5), (2, 6), (3, 6)$. The number of committees for each of these 4 cases is $\binom{4}{2}\binom{6}{4}, \binom{4}{2}\binom{6}{5}, \binom{4}{2}\binom{6}{6}, \binom{4}{3}\binom{6}{6}$, respectively, the total of which is 136.

b

The number of committees of size between 3 and 5 that can be formed from 4 men and 6 women when 1 woman is excluded is $\binom{9}{3} + \binom{9}{4} + \binom{9}{5} = 336$, since we must choose 3, 4 or 5 people from the remaining 9 candidates.

c

The number of committees of size 5 that can be formed from 10 people but not containing a specific group of 3 people is $\binom{10}{5} - \binom{7}{2} = 231$ since there are $\binom{10}{5}$ subsets of size 5 and $\binom{7}{2}$ subsets of size 5 containing a specific subset of size 3.

d

To count the number of committees of size 4 that can be formed from 4 men and 6 women where ≥ 2 women and not containing a specific man, woman pair, note that the number of men, women chosen must be one of $(2, 2), (1, 3), (0, 4)$. The number of committees for each case without the restriction that the specific man, woman pair is excluded is $\binom{4}{2}\binom{6}{2}, \binom{4}{1}\binom{6}{3}, \binom{4}{0}\binom{6}{4}$. To form a committee of size 4 with at least 2 women and containing the specific man, woman pair, we must select the specific man, woman pair and then the number of extra men, women must be one of $(1, 1), (0, 2)$. The number of committees for each case is $\binom{3}{1}\binom{5}{1}, \binom{3}{0}\binom{5}{2}$.

So the number of committees of size 4 with at least 2 women and excluding the specific man, woman pair is $\binom{4}{2}\binom{6}{2} + \binom{4}{1}\binom{6}{3} + \binom{4}{0}\binom{6}{4} - ((\binom{3}{1}\binom{5}{1}) + (\binom{3}{0}\binom{5}{2})) = 160$.

5.2.27

The smallest n such that there are at least 365 subsets of $\{1, \dots, n\}$ of size 4 is $\min\{n \geq 0 \mid \binom{n}{4} \geq 365\} = 12$, which was easily be found by binary search since $\binom{n}{4}$ is an increasing function of n .

5.2.41

a

To choose a 5-card hand so that at least 1 of each of the 4 face values appears, we may choose one of the 4 face values to appear twice and the other 3 to appear once each, or we may choose each of the 4 face values to appear once and a single non-face value to appear. So the number of such hands is $4 \cdot \binom{4}{2}\binom{4}{1}^3 + \binom{4}{1}^4 \binom{52-4 \cdot 4}{1}$. The probability of choosing such a hand is then

$$\frac{4 \cdot \binom{4}{2}\binom{4}{1}^3 + \binom{4}{1}^4 \binom{52-4 \cdot 4}{1}}{\binom{52}{5}} = \frac{32}{7735} \approx .004.$$

b

To calculate the probability that a randomly selected 5-card hand has the same number of hearts as spades, note that the number of hearts, spades, others chosen must be one of $(0, 0, 5), (1, 1, 3), (2, 2, 1)$. The number of such cases is $\binom{13}{0}^2 \binom{26}{5}, \binom{13}{1}^2 \binom{26}{3}, \binom{13}{2}^2 \binom{26}{1}$, respectively. So the desired probability is

$$\frac{\binom{13}{0}^2 \binom{26}{5} + \binom{13}{1}^2 \binom{26}{3} + \binom{13}{2}^2 \binom{26}{1}}{\binom{52}{5}} = \frac{12757}{49980} \approx .26.$$

1 Exercises from Bender

1.5

The number of sequences of letters from $a, \dots, z$ of length 4 containing at least one of a, e, i, o, u is the number of sequences of letters from $a, \dots, z$ of length 4 minus the number of such sequences that fail to contain any of a, e, i, o, u ; i.e. $26^4 - (26 - 5)^4 = 262495$.

2.1

a

The number of ways to arrange 3 (distinct) boys and 4 (distinct) girls is $7! = 5040$.

b

The number of ways to arrange 3 boys and 4 girls such that the boys are consecutive and the girls are consecutive is $2 \cdot 3! \cdot 4! = 288$, since either the boys come first or the girls, and then there are $3!$ ways of arranging the boys among themselves and $4!$ ways of arranging the girls among themselves.

c

The number of ways to arrange 3 boys and 4 girls such that the gender alternates is $3! \cdot 4! = 144$ since these sequences are in one-to-one correspondence with the permutations of 4 girls and the permutations of 3 boys. The correspondence is to look at the girls in positions 1, 3, 5, 7 and then the boys in positions 2, 4, 6.

2.2

a

The number of ways to arrange 3 (distinct) boys and 3 (distinct) girls is $6! = 720$.

b

The number of ways to arrange 3 boys and 3 girls such that the boys are consecutive and the girls are consecutive is $2 \cdot 3! \cdot 3! = 72$, since either the boys come first or the girls, and then there are $3!$ ways of arranging the boys among themselves and $3!$ ways of arranging the girls among themselves.

c

If the girls and boys must alternate, then either the girls occupy positions 1, 3, 5 or they occupy positions 2, 4, 6. So the number of such arrangements is $2 \cdot 3! \cdot 3! = 72$.

2.7

a

The sequences of letters from R,L,F,I,E of length ≤ 6 that contain ≥ 1 L, begin with R, end with F, contain exactly one R and exactly one F must either have 1, 2, 3 or 4 letters between the R and the F. The number of such sequences with i letters between the R and the F is $3^i - 2^i$ since there are 3^i sequences of letters from L,F,I of length i and 2^i of them avoid using the L. So the number of sequences of letters from R,L,F,I,E of length ≤ 6 that contain ≥ 1 L, begin with R, end with F, contain exactly one R and exactly one F is $3^1 - 2^1 + 3^2 - 2^2 + 3^3 - 2^3 + 3^4 - 2^4 = 90$.

b

The 5 sequences preceding RELIEF in dictionary order (going backwards) are

1. RELELF
2. RELEIF
3. RELEEF
4. REILLF
5. REILIF.