

CSE 21—Mathematics for Algorithm and System Analysis

Summer, 2005

Day 8
Generating Functions

Instructor: Neil Rhodes

Outline

What a generating function is
How to create a generating function to model a problem
Finding the desired coefficient
Partitions
Exponential generating functions

2

(Ordinary) Generating Functions

A different way to solve problems involving selection and arrangement problems with repetition

Can easily support special constraints not easy to do otherwise

- Find the number of ways to distribute 70 identical objects into 9 boxes with an odd number between 3 and 7 in the first box, an even number between 4 and 20 in the second box, and at most 3 in the other boxes.

Idea

- Create a generating function $g(x)$ which models the problem of selecting r objects. The coefficient of x^r in $g(x)$ counts the number of ways to select r objects
- $g(x) = a_0 + a_1x + a_2x^2 + \dots + a_rx^r + \dots + a_nx^n$

Example

- $g(x) = (1+x)^n$ models the number of ways to select r objects from n objects

$$g(x) = (1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n$$

3

Let's look at the coefficients of x^5 in $(1+x+x^2)^4$

- Ways to create x^5 :
 - $x^0x^2x^2x^1$
 - $x^1x^1x^1x^2$
 - etc.
- In general, $x^{e_1}x^{e_2}x^{e_3}x^{e_4}$
 - where $0 \leq e_1, e_2, e_3, e_4 \leq 2$
- Can be looked at as the number of integer solutions to:
 - $e_1 + e_2 + e_3 + e_4 = 5$ (where $0 \leq e_1, e_2, e_3, e_4 \leq 2$)
- Which can also be looked at as the number of ways to place 5 balls in four boxes with at most 2 balls in each box
- Which can also be looked at as the number of ways to select 5 objects from a collection of 4 different types of objects (with two identical objects of each type).

$g(x) = (1+x+x^2)^4$ is a generating function for the above problems

- Actually, $g(x)$ is a generating function for the problem with any number, r , of {balls, objects}, not just 5
- We always write a general generating function for any r

4

Example

Find a generating function for a_r , the number of ways to select r balls from a pile of three green, three white, three blue, and three gold balls

- Same as number of integer solutions to:
- $e_1 + e_2 + e_3 + e_4 = r \quad 0 \leq e_i \leq 3$

5

Example

Find a generating function to model counting all selections of six objects chosen from three types of objects with:

- repetition of up to four objects of each type

- unlimited repetition

6

Example

Find a generating function to model:

- the number of ways to distribute r identical objects into five boxes with an even number of objects not exceeding 10 in the first two boxes and between 3 and 5 in the other boxes.

7

Example

Find a generating function for a_r , the number of ways a roll of six distinct dice can show a sum of r if

- The first three dice are odd and the second three even

- The i th die does not show a value of i

8

Finding the Desired Coefficient

Reduce complicated generating function to simple function or product of simple functions

- The simple functions:

$$\frac{1-x^{m+1}}{1-x} = 1+x+x^2+\dots+x^m$$

$$\frac{1}{1-x} = 1+x+x^2+\dots$$

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n$$

$$(1-x^m)^n = 1 - \binom{n}{1}x^m + \binom{n}{2}x^{2m} - \dots + (-1)^k \binom{n}{k}x^{km} + \dots + \binom{n}{n}x^{nm}$$

$$\frac{1}{(1-x)^n} = 1 + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \dots + \binom{n-1}{r}x^r + \dots$$

9

Product of Generating Functions

If

- $f(x) = a_0 + a_1x + a_2x^2 + \dots$
- $g(x) = b_0 + b_1x + b_2x^2 + \dots$

and

- $h(x) = f(x)g(x)$

then

- $h(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots + (a_0b_r + a_1b_{r-1} + \dots + a_{r-1}b_1 + a_rb_0)x^r + \dots$

10

Example

Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \dots)^5$

- Rewrite as product of simpler functions

- Final simplification:

- Find coefficient

11

Example

Find the number of ways to collect \$15 from 20 distinct people if each of the first 19 people can give \$0 or \$1 and the 20th person can give either \$0 or \$1 or \$5

- Generating function: $g(x) =$

- Simplify

- Find coefficient

12

Example

How many ways to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls?

- Generating function
- Simplify
- Find coefficient

13

Example

How many ways are there to select 25 toys from seven types of toys with between two and six of each type?

- Generating function
- Simplify
- Find coefficient

14

Partitions

Definition

- A partition of r identical objects divides the group into a collection (unordered) subsets of various sizes
- (Or, a group of positive integers whose sum is r)

Example

- Partitions of 5

We'll be looking at creating generating functions for partitions

Warning

- No easy way to calculate the coefficients

15

Generating Function for Partition

Generating function for a_r

- Want to know
 - e_1 : how many 1's
 - e_2 : how many 2's
 - ...
- $1e_1 + 2e_2 + \dots + re_r = r$
- Choose ones:
 - $(1 + x + x^2 + x^3 + \dots + x^n + \dots) = 1/(1-x)$
- Choose twos
 - $(1 + x^2 + x^4 + \dots + x^{2n} + \dots) = 1/(1-x^2)$
- Choose threes
 - $(1 + x^3 + x^6 + \dots + x^{3n} + \dots) = 1/(1-x^3)$
- and so on.

$g(x)$ = product of all those

- $= 1/(1-x)(1-x^2)(1-x^3)\dots(1-x^r)\dots$

16

Example

Find a generating function for the number of ways to express r as a sum of distinct integers

17

Example

Show with generating functions that every integer can be expressed as a unique sum of distinct powers of 2

- Generating function

- To show: $g(x) = 1 + x + x^2 + x^3 + \dots$

18

Ferrers Diagram

Graphically displays a partition of r dots

- In a set of rows of decreasing size

- Conjugate: Rotate 90 degrees counter-clockwise

19

Example

Show that the number of partitions of an integer r as a sum of m positive integers is equal to the number of partitions of r as a sum of positive integers, the largest of which is m

20

Exponential Generating Functions

Used to model problems involving arrangements and distributions of distinct objects

Example problem

- find number of different arrangements of four letters, choosing from a's, b's, and c's with at least two a's.

Solution

- Four letter combinations:
 - {a, a, a, a}: $4!/4!0!0!$
 - {a,a,a,b}: $4!/3!1!0!$
 - {a,a,a,c}: $4!/3!0!1!$
 - {a,a,b,b}: $4!/2!2!0!$
 - {a,a,b,c}: $4!/2!1!1!$
 - {a,a,c,c}: $4!/2!0!1!1!$
- Total is sum, so we want that sum as the coefficient of x^4
- Ordinary generating function can solve number of integer solutions to:
 - $e_1 + e_2 + e_3 = 4$ $e_1 \geq 2$ $e_2, e_3 \geq 0$
- But, we want each solution to above problem to contribute:
 - $(e_1+e_2+e_3)!/e_1!e_2!e_3!$

21

Exponential Generating Function

Function of form:

- $g(x) = a_0 + a_1x + a_2x^2/2! + a_3x^3/3! + \dots + a_rx^r/r! + \dots$

Example

- $g(x) = (x^2/2! + x^3/3! + x^4/4! + \dots)(1+x+x^2/2! + x^3/3! + \dots)^2$
- Coefficient of x_r in $g(x)$ is equal to: $\sum_{e_1+e_2+e_3=r} \frac{1}{e_1!e_2!e_3!}$
- Coefficient of $x_r/r!$ in $g(x)$ is equal to: $\sum_{e_1+e_2+e_3=r} \frac{r!}{e_1!e_2!e_3!}$

22

Example

Find the exponential generating function for a_r , the number of different arrangements of r objects chosen from four different type of objects with each type of object appearing at least 2 and no more than 5 times

23

Example

Find the number of ways to place r (distinct) people into three rooms with at least one person in each room

24

Finding Desired Coefficients

Useful identities

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$$

$$e^{nx} = 1 + nx + \frac{n^2x^2}{2!} + \frac{n^3x^3}{3!} + \cdots + \frac{n^rx^r}{r!} + \cdots$$

$$\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots +$$

$$\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots +$$

25

Example

Find the number of different r arrangements of objects chosen from unlimited supplies of n types of objects

- Generating function

- Simplify

- Find coefficient

26

Example

Find the number of ways to place 25 people into 3 rooms with at least one person in each room

- Generating function

- Simplify

- Find coefficient

27

Example

Find the number of r -digit quaternary sequences (digits: 0, 1, 2, 3) with an even number of 0's and an odd number of 1's

- Generating function

- Simplify

- Find coefficient

28