

CSE 21—Mathematics for Algorithm and System Analysis

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Day 7
Recurrence Relations

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Motivating Example

Create a recursive formula to specify how many ways to climb an n -stair staircase if each step covers either one or two stairsteps

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Recurrence Relation

A recursive formula counting the number of ways to do something with n objects in terms of the number of ways to do it with fewer objects

- Example

- $a_n = a_{n-1} + a_{n-2}$

General format

- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r}$
- $a_n = c a_{n-1} + f(n)$
- $a_n = a_0 a_{n-1} + a_1 a_{n-2} + \dots + a_{n-1} a_0$
- $a_{n,m} = a_{n-1,m} + a_{n-1,m-1}$

Must also have *initial conditions*

- Example

- $a_0 = 0, a_1 = 3$

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Example

A boy has n cents. Each day, he can buy one of:

- a candy bar (for 20 cents)
- a piece of gum (for 1 cent)
- an apple (for 10 cents)

How many ways are there he can buy items until the money is gone?

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Example

How many n -digit ternary sequences without any occurrence of the subsequence 012?

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Difference Equations

Sometimes easier to look at differences rather than absolutes

- Foxes/Rabbits
 - Rabbits (r_n) increase each year by $a_0 r_n$ (reproduce), but foxes eat them at the rate $a_1 r_n f_n$
 - Foxes alone decrease each year by $a_1 f_n$, but with rabbits to eat, increase by rate $a_3 r_n f_n$
 - $\Delta r_n = a_0 r_n - a_1 r_n f_n$
 - $\Delta f_n = a_3 r_n f_n - a_2 f_n$
- Differential equations (continuous) $\approx$ Difference equations (discrete)

First backward difference $\Delta a_n = a_n - a_{n-1}$

Second backward difference $\Delta^2 a_n = \Delta a_n - \Delta a_{n-1}$

- Example: first and second differences of n^2

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Solving a Recurrence Relation

Bottom-up method

- Given a particular n (not too big), use the recurrence relation (and initial conditions) to successively find $a_0, a_1, \dots, a_n$

Substitution method

- Substitute solution for a_{n-1} , then a_{n-2} , etc.
 - Example: $a_n = n a_{n-1}$

Guess and Verify

- Guess a solution
- Use induction to verify
 - Example: $a_n = 1.08 a_{n-1}, a_0 = 100$
 - Guess: $a_n = 100(1.08)^n$

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Solving a Recurrence Relation

Given a recurrence relation of form

- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r}$ linear homogeneous with constant coefficients

Rewrite as powers:

- $\alpha^n = c_1 \alpha^{n-1} + c_2 \alpha^{n-2} + \dots + c_r \alpha^{n-r}$

Simplify

- $\alpha^r - c_1 \alpha^{r-1} - c_2 \alpha^{r-2} - \dots - c_r = 0$ characteristic equation

Solve for r roots, $\alpha_1, \dots, \alpha_r$

Generate form of final solution ($a_n = \alpha_i^n$ is a solution to rec. relation)

- $a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \dots + A_r \alpha_r^n$

Use initial conditions to create a set of simultaneous equations

- $a_0 = A_1 \alpha_1^0 + A_2 \alpha_2^0 + \dots + A_r \alpha_r^0$
- ...
- $a_{r-1} = A_1 \alpha_1^{r-1} + A_2 \alpha_2^{r-1} + \dots + A_r \alpha_r^{r-1}$

Solve for $A_1, \dots, A_r$

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Example

Give closed form for:

- $a_n = 1.08a_{n-1}$
- $a_0 = 500$

Rewrite as powers

- $\alpha^n = 1.08\alpha^{n-1}$

Simplify

- $\alpha - 1.08 = 0$

Solve for α_1

- $\alpha_1 = 1.08$

Generate form of final equation

- $a_n = A_1 \alpha_1^n$

Use initial conditions

- $a_0 = 500 = A_1$

Final equation

- $a^n = 500(1.08)^n$

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Example

Find a closed form for the number of ways to make a pile of n chips using red, white and blue chips and so that no two red chips are together

- $a_n =$
- $a_0 =$
- $a_1 =$

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Example

Find a closed form for the Fibonacci sequence

- $a_n = a_{n-1} + a_{n-2}$
- $a_0 = a_1 = 1$

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Adjustment for Multiple Roots

Given a recurrence relation of form

- $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_r a_{n-r}$

...

Solve for r roots, $\alpha_1, \dots, \alpha_r$

If we have a root α_i of multiplicity 2, general form of equation is

- $a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \dots + A_i \alpha_i^n + A_{i+1} n \alpha_i^n + \dots + A_r \alpha_r^n$

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Example with Multiple Roots

Given

- $a_n = -2a_{n-2} - a_{n-4}$
- $a_0 = 0$
- $a_1 = 1$
- $a_2 = 2$
- $a_3 = 3$

Find a closed form for a_n

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Divide & Conquer Recurrence Relations

If there are n (a power of two) players in a single-elimination tournament, how many rounds need to be played?

- $a_n = a_{n/2} + 1$

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Solving Recurrence Relations

Given

- $a_n = ca_{n/k} + f(n)$

Three general cases (in the long run):

- $f(n)$ grows faster than $n^{\log_k c}$
 - a_n grows proportional to $f(n)$
- $f(n)$ grows slower than $n^{\log_k c}$
 - a_n grows proportional to $n^{\log_k c}$
- $f(n)$ grows proportional to $n^{\log_k c}$
 - a_n grows proportional to $\log_k n \cdot n^{\log_k c}$

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Example

Merge-sort algorithm for list of n (power of 2) items:

- break list into two equal parts: L1, L2
- recursively sort L1 and L2
- Merge L1 and L2 together into L (takes $2n-1$ operations)

Recurrence relation

- $a_n =$

How fast does it grow?

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Solving a Recurrence Relation

Given a recurrence relation of form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r} + f(n)$$

linear inhomogeneous with constant coefficients

Remove $f(n)$ and find general solution:

$$a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \dots + A_r \alpha_r^n$$

Use table to find form of particular solution for inhomogenous eqn.

$$a^* n = p(n) = c_1 p(n-1) + \dots + c_r p(n-r) + f(n)$$

Solve for coefficient(s) in $p(n)$

Modify general solution by adding $p(n)$

$$a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \dots + A_r \alpha_r^n + p(n)$$

Solve for coefficient(s)

$f(n)$	form of $P(n)$
d	B
dn	$B_1 n + B_0$
dn^2	$B_2 n^2 + B_1 n + B_0$
ed^n	Bd^n

Special case: $c=1, r=1$

$$a_n = \sum f(n)$$

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Example

Find closed-form solution for

$$\begin{aligned} a_n &= a_{n-1} + n \\ a_0 &= 1 \end{aligned}$$

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Example

Find closed form for

$$\begin{aligned} a_n &= 2a_{n-1} + 1 \\ a_1 &= 1 \end{aligned}$$

General form for homogenous part

$$a_n = A2^n$$

Form of particular solution

$$\begin{aligned} p(n) &= B \\ a^* n = p(n) &= B = 2a^*_{n-1} + 1 = 2B + 1 \\ B &= -1 \end{aligned}$$

General form with particular solution added:

$$\begin{aligned} a_n &= A2^n - 1 \\ A &= 1 \end{aligned}$$

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Example

Find closed form for

$$a_n = 3a_{n-1} - 4n + 3 \cdot 2^n$$

General form of homogenous part

$$a_n = A3^n$$

Form of first inhomogenous term

$$\begin{aligned} p(n) &= B_1 n + B_0 \\ p(n) = B_1 n + B_0 &= a^*_n = 3a^*_{n-1} - 4n = 3(B_1(n-1) + B_0) - 4n \\ &= \end{aligned}$$

Form of second inhomogenous term

$$\begin{aligned} q(n) &= B \cdot 2^n \\ q(n) = B \cdot 2^n &= a^*_n = 3a^*_{n-1} + 3 \cdot 2^n = 3B \cdot 2^{n-1} + 3 \cdot 2^n \\ &= \end{aligned}$$

Final general solution

$$a_n = A3^n + p(n) + q(n)$$

Final solution

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Example

Given:

- the average of 2 successive years' production $\frac{1}{2}(a_n + a_{n-1})$ is $2n + 5$
- $a_0 = 3$

Find a_n