

CSE 21—Day 5

Functions/Probability

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Overview

- Functions
- Probabilistic analysis

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Image/Coimage

- The *image* of f is the set of values f actually takes on (a subset of the codomain)
- The *inverse image*, $f^{-1}(b)$ of an element in the image is the set of values in the domain that map to b .
- Given $f : A \rightarrow B$, the *coimage* of f is the collection of nonempty inverse images of elements of B
- The coimage of A is a partition of A .
- f^{-1} is a function with domain B and codomain the powerset of A

Monotonic Lists

- Want unique ordering of k -multisets whose elements lie in $\{1 \dots n\}$
- Can use weakly increasing ordered k -lists made from $\{1 \dots n\}$
- Bijection between:
 - Weakly-increasing ordered k -lists made from $\{1 \dots n\}$
 - k -multisets whose elements lie in $\{1 \dots n\}$
 - Weakly increasing functions
 $f \in \{1 \dots n\}^{\{1 \dots k\}}$
- Example: 3-multisets from $\{1..5\}$

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Monotonic Lists

- Bijection between:
 - Strictly-increasing ordered k -lists made from $\{1 \dots n\}$
 - k -sets whose elements lie in $\{1 \dots n\}$
 - Strictly increasing functions $f \in \{1 \dots n\}^{\{1 \dots k\}}$
- Example: 3-sets from $\{1 \dots 7\}$

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Monotonic Lists

- How many ways to put five identical balls into four distinct boxes?
- Describe a systematic way to list the configurations and give the first 3 and last 3 configurations in the list.

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Probability

- A *sample space* S is a set of *elementary event*
 - Rolls of a die = $\{1, 2, 3, 4, 5, 6\}$ or flipping two coins = $\{HH, HT, TH, TT\}$
- A *event* is a subset of S .
 - Rolling better than a 4 = $\{5, 6\}$
 - Flipping at least one head = $\{HH, HT, TH\}$
- A *probability distribution* Pr on a sample space is a function mapping events to real numbers $\in [0..1]$ such that:
 - $Pr\{S\} = 1$
 - $Pr\{A \cup B\} = Pr\{A\} + Pr\{B\}$ for mutually exclusive events

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Probability

- If S is finite or countably infinite, then the probability distribution is *discrete*.
- If S is finite and each elementary event $e \in S$ has $Pr\{e\} = 1/|S|$, then we have a *uniform probability distribution*.
 - Probability distribution for rolling dice $\{1, 2, 3, 4, 5, 6\}$ is uniform
 - Probability distribution for scores on the SAT $\{400, 410, \dots, 1590, 1600\}$ is not uniform.

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Random variable

- A *(discrete) random variable* X is a function from the elementary events in a (finite or countably infinite) sample space S to the real numbers. Examples:
 - Flip k coins (sample space has 2^k elementary events). X is the total number of heads flipped ($0 \leq X(e) \leq k$)
 - Roll two fair dice. $S = \{(1, 1), (1, 2), \dots, (6, 6)\}$. Y is the total shown on the dice

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Probability

- The event $X = x$ is defined to be $\{s \in S : X(s) = x\}$.
- The function $f(x) = \Pr\{X = x\}$ is the *probability density function*.
 - Roll a pair of dice (36 elementary events, uniform probability distribution).
 - Define X to be the maximum of the two values.
 - $\Pr\{X = 3\} = 5/36$

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Multiple Random Variables

- If X, Y random variables:
 - $f(x, y) = \Pr\{X = x \& Y = y\}$ is the *joint probability density function* of X and Y .
 - X and Y are *independent* if, for all x and y $f(x, y) = f(x)f(y)$.
 - $X + Y$ is a function. For example, $(X + Y)(e) = X(e) + Y(e)$

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Expectation

- The *Expectation* $E[X]$ is $\sum_{e \in S} X(e) \Pr\{e\}$. This is, the weighted average of X .
 - Throw a die. Y is the total shown on the die. $E[Y] = 1 \cdot \Pr\{1\} + 2 \cdot \Pr\{2\} + \dots + 6 \cdot \Pr\{6\} = 3.5$
 - Pick a person from this classroom. X is the age of that person $E[X] \approx 1 \cdot \Pr\{18\} + 4 \cdot \Pr\{19\} + \dots + 1 \cdot \Pr\{43\} =$

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Linearity of Expectation

- *Linearity of Expectation* theorem:
 $E[aX + bY] = aE[X] + bE[Y]$ (regardless of whether X and Y are dependent or independent)
 - Throw two dice. $Y + Y$ is the total shown.
 $E[Y + Y] = E[Y] + E[Y] = 2E[Y] = 2 \cdot 3.5$
 - Add together 2 things: (b) the total of five dice and (b) twice the age of a person chosen uniformly at random from this classroom
 $2X + 5Y$ is the total. $E[2X + 5Y] = 2E[X] + 5E[Y] = 45 + 17.5 = 62.5$

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Indicator Random Variables

- An *indicator random variable* $I(A)$ associated with event A is 1 if A occurs and 0 otherwise
- The expected value of an indicator random variable is the probability the associated event occurs.
 - Let $Y_i = I(\text{a die roll shows } i)$

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Example: Indicator Random Variable

- Flip a coin n times. What is the expected number of heads?
- Analysis using expectations of indicator random variables.
 - Let $X_i = I\{\text{the } i\text{th flip results in the event}\}$
 - Let X be the random variable denoting the total number of heads in the n coin flips.
 - $X = \sum_{i=1}^n X_i$

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Example: Hiring Problem

- Candidates appear in random order (all permutations equally likely)
- If candidate is better than best seen so far, hire candidate
- How many hirings do we expect to do?
- Analysis using expectations of indicator random variables.
 - Let $X_i = I\{\text{the } i\text{th candidate is hired}\}$.
 - Let X be the random variable denoting the total number of candidates hired.
 - $X = \sum_{i=1}^n X_i$

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Example: Hat Check Problem

- n customers give hat to hat-check clerk
- Hats given back in random order (all permutations equally likely)
- What is expected number of people who get their own hat back?
- Analysis using expectations of indicator random variables.
 - Let $X_i = I\{\text{the } i\text{th person gets his hat back}\}$
 - Let X be the random variable denoting the total number of people getting their hat back
 - $X = \sum_{i=1}^n X_i$

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Example: Birthday Problem

- How many people must be in a room so that the expected number of matching birthdays is 1?
 - Let $X_{ij} = I\{\text{Person } i \text{ and } j \text{ have the same birthday}\}$.
 - Let X be the random variable denoting the number of pairs of individuals sharing the same birthday.
 - $X = \sum_{i=1}^n \sum_{j=i+1}^n X_{ij}$

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Example: Birthday Problem

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Measuring Variability

- *Variance* of a random variable X is:

$$\begin{aligned}\text{Var}[X] &= \sigma_x^2 = E[(X - E[X])^2] \\ &= E[X^2] - E^2[X]\end{aligned}$$

- *Standard Deviation*, σ_X of a random variable is the positive square root of the variance, σ_X^2 .
- *Covariance* of two random variables, X and Y
 $\text{Cov}[X, Y] = E[XY] - E[X]E[Y]$
- *Correlation* of two random variables, X and Y
 $\rho(X, Y) = \text{Cov}[X, Y] / \sigma_X \sigma_Y$ (if $\sigma_X, \sigma_Y \neq 0$)

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Correlation

- Correlation ranges from -1 to 1.
 - Near 1 means events where X is large tend to have Y large too.
 - Near 0 means no linear relationship
 - Near -1 means events where X is large tend to have Y small.
- $\text{Var}[aX + bY + c] = a^2\text{Var}[X] + 2ab\text{Cov}[X, Y] + b^2\text{Var}[Y]$

Tchebycheff's Inequality

- Bounds the probability that a random variable is at least a away from the mean, μ_X :

$$\Pr\{|X - \mu_X| \geq a\} \leq \frac{\sigma_X^2}{a^2}$$

- Example, for any distribution function of X , the probability that X is at least 3 standard deviations from the mean is:

Geometric Distribution

- A *Bernoulli trial* is an experiment with 2 outcomes: success with probability p , and failure with probability $q = 1 - p$
- Assume a sequence of n independent Bernoulli trials.
- How many trials before we get a success?
 - Let X be the number of trials needed to obtain a success
 - The probability distribution $\Pr\{X = k\} = q^{k-1}p$ is a *geometric distribution*
 - $E[X] = 1/p$
 - $\text{Var}[X] = q/p^2$

Binomial Distribution

- How many successes during n Bernoulli trials
- Let X be the number of successes in n trials
- The probability distribution $\Pr\{X = k\} = \binom{n}{k} p^k q^{n-k}$ is a *binomial distribution*
- $E[X] = np$
- $\text{Var}[X] = npq$

Normal Distribution

- Standard bell curve, or Gaussian distribution. Used with the sum of many (nearly) independent random variables
- Parameterized based on μ and σ .

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

- if mean is 0 and standard deviation is 1, we have the *standard normal distribution*:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

- Binomial distribution is approximately normal for large n and not too close to 0 or 1

Standard Normal Distribution

- $\Pr\{0 \leq X \leq 1\} \approx .34$
- $\Pr\{0 \leq X \leq 1.5\} \approx .43$
- $\Pr\{0 \leq X \leq 2\} \approx .48$
- Converting from normal distribution to Standard Normal Distribution:
 - Normalize by subtracting mean and dividing by standard deviation
 - Example: oranges with mean weight of 8 ounces and standard deviation of .67 ounce (normal distribution). What fraction weigh at least 9 ounces?

Standard Normal Distribution

- 400 words on a page with about one word in 10 wrong. What is the probability of at most 30 errors?
 - Binomial distribution with $p = .1$
 - Approximate with normal distribution:
 $\mu = 40, \sigma = \sqrt{400 \cdot .1 \cdot .9} = 6$

Central Limit Theorem

- Let $X_1, \dots, X_n$ be a sequence of independent, identically distributed random vars, each with mean μ and variance σ^2 . Let $S_n = X_1 + X_2 + \dots + X_n$.
 - $E[S_n] = n\mu$
 - $\sigma_{S_n} = \sigma n^{1/2}$
- Theorem: As $n \rightarrow \infty$, the distribution of $S_n \rightarrow$ standard normal.