

CSE 21—Mathematics for Algorithm and System Analysis

Summer, 2005

Day 3
Binomial Coefficients
Inclusion-Exclusion
Probability

Instructor: Neil Rhodes

All of These are Equivalent

1. The number of ways to select r objects with repetition from n different types of objects
2. The number of ways to distribute r identical objects into n distinct boxes
3. The number of nonnegative integer solutions to:
 - $x_1 + x_2 + \dots + x_n = r$.

2

Summary

Arrange/Select/Distribute r objects from n items

	Arrangement (ordered outcome) or Distribution of distinct objects	Combination (unordered outcome) or Distribution of identical objects
No repetition		
Unlimited repetition		
Restricted repetition		

3

Binomial Coefficients

Binomial Theorem

Let's look at $(x+y)^3$

- $(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

- How many terms of the form x^ky^{3-k} in $(x+y)^3$?

How many ways to choose k **x**s and $(3-k)$ **y**s from 3 total?

- $C(3, k)$

5

Binomial Theorem

Let's look at $(x+y)^n$

- What is the coefficient of x^ky^{n-k} in $(x+y)^n$?

$$\binom{n}{k}$$

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{k}x^{n-k}y^k + \dots \binom{n}{n}y^n$$

6

Binomial Identities

Symmetry Identity

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Fundamental Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Algebraic Proof
- Proof by Pascal's triangle
- Proof by combinatorial argument

7

Pascal's Triangle

Number of paths equals $C(n, k)$

8

Inclusion/Exclusion

Venn Diagrams

Set A in universe U

10

Venn Diagrams

Sets L and F in universe U

$$|L \cup F| = |L| + |F| - |F \cap L|$$

L = # students taking Latin

F = # students taking French

11

Example (from 8.1 example 2)

How many arrangements of the digits 0,1,...,9 are there in which the first digit is greater than 1 and the last digit is less than 8?

12

Venn Diagram with Three Sets

Sets F, L, G in universe U

Number students not taking any language?

13

Inclusion-Exclusion Formula for 3 Sets

$$N(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3) =$$

$$\begin{aligned} & N \\ & - [N(A_1) + N(A_2) + N(A_3)] \\ & + [(N(A_1 \cap A_2) + N(A_2 \cap A_3) + N(A_3 \cap A_1))] \\ & - N(A_1 \cap A_2 \cap A_3) \\ & = N - \sum_i N(A_i) + \sum_{i,j} N(A_i \cap A_j) - N(A_1 \cap A_2 \cap A_3) \end{aligned}$$

14

Example (from 8.1, example 4)

How many positive integers ≤ 70 are relatively prime to 70?

15

Example (from 8.1, exercise 31)

Suppose a school with 120 students offers Yoga and Karate. If the number of students taking Yoga is twice the number taking Karate (possibly, Karate and Yoga), if 25 more students study neither skill than study both skills, and if 75 students take at least one skill, then how many students study Yoga?

16

Inclusion-Exclusion Formula for n Sets

Let $A_1, A_2, \dots, A_n$ be sets in a universe U of N elements. Let S_k denote the sum of the sizes of all k -tuple intersections of the A_i s.

$$S_1 = \sum_i N(A_i)$$

$$S_2 = \sum_{i,j} N(A_i \cap A_j)$$

S_k = sum of all intersections of exactly k A_i s.

$$S_n = N(A_1 \cap A_2 \dots \cap A_n)$$

17

Inclusion-Exclusion Formula for n Sets

Let $A_1, A_2, \dots, A_n$ be sets in a universe U of N elements. Let S_k denote the sum of the sizes of all k -tuple intersections of the A_i s.

$$N(\bar{A}_1 \cap \bar{A}_2 \dots \cap \bar{A}_n) = N - S_1 + S_2 - S_3 + \dots + (-1)^k S_k + \dots + (-1)^n S_n$$

- number of times an element is counted which is in exactly m A_i s:

$$1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \dots + (-1)^k \binom{m}{k} + \dots + (-1)^m \binom{m}{m}$$

- $(1+x)^m$

Corollary:

$$N(A_1 \cup A_2 \dots \cup A_n) = S_1 - S_2 + S_3 - \dots + (-1)^{k-1} S_k + \dots + (-1)^{n-1} S_n$$

18

Example (from 8.2 example 1)

How many ways are there to select a 5-card hand from a regular 52-card deck such that the hand contains at least one card in each suit?

- A_1 = void in clubs
- A_2 = void in hearts
- A_3 = void in spades
- A_4 = void in diamonds
- Calculate $N(\bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3 \cup \bar{A}_4)$
- $N - S_1 + S_2 - S_3 + S_4$

19

Example (from 8.2, example 3)

How many different integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20 \quad (0 \leq x_i \leq 8)$$

- A_i = set of integer solutions with $x_i > 8$

20

Example (from 8.2, example 2)

How many ways are there to distribute r distinct objects into five (distinct) boxes with at least one empty box?

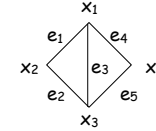
- Let A_i = set of distributions with a void in box i .

21

Example (from 8.2, example 5)

How many ways are there to color the four vertices in a graph with n colors (such that the vertices with a common edge must be different colors)?

- Define A_i = all colorings in which either end of edge e_i have the same color.



- $N =$
- $S_1 =$
- $S_2 =$
- $S_3 =$
- $S_4 =$
- $S_5 =$
- Total =

22

Probability

Definitions

A *sample space*, S , is a set of *elementary events*

- Rolls of a die = $\{1, 2, 3, 4, 5, 6\}$
- Flipping two coins = $\{HH, HT, TH, HH\}$

An *event* is a subset of a sample space:

- Rolling better than a 4 = $\{5, 6\}$
- Flipping at least one head = $\{HH, HT, TH\}$

A *probability distribution* (or *probability function*), P , is a function which assigns a real value to each event t in S such that:

- $P(t) \geq 0$
- The sum over all t of $P(t)$ equals 1
- For an event E , $P(E)$ = the sum of the probabilities of each elementary event in E
 - Probability of rolling better than a 4 is $P(5) + P(6) = 2/6$
 - Probability of flipping at least one head is $P(HH) + P(HT) + P(TH) = 3/4$

A *probability space* is a combination of a sample space and a probability distribution

If the sample space S is finite and for each elementary event e , $P(e) = 1/|S|$, then P is a *uniform probability distribution*

- Probability distribution for rolling dice $\{1, 2, 3, 4, 5, 6\}$ is uniform
- Probability distribution for scores on the SAT $\{400, 410, \dots, 1590, 1600\}$ is not uniform

24

Theorem of Union of Events

If (S, P) is a sample space and A and B are disjoint subsets of S , then:

- $P(A \cup B) = P(A) + P(B)$
- For rolling dice, if $A = \{5, 6\}$ and $B = \{1, 2\}$, then $P(\{1, 2, 5, 6\}) = 2/3$

In general, (A and B may or may not be disjoint):

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- For rolling dice, if $A = \{5, 6\}$ and $B = \{2, 4, 6\}$, then
$$\begin{aligned} P(\{5, 6\} \cup \{2, 4, 6\}) &= P(\{5, 6\}) + P(\{2, 4, 6\}) - P(\{5, 6\} \cap \{2, 4, 6\}) \\ &= 1/3 + 1/2 - P(\{6\}) \\ &= 1/3 + 1/2 - 1/6 \\ &= 2/3 \end{aligned}$$

25

Example (from CL exercise 4.4)

Six horses are in a race. You pick two of them at random and bet on them both. Find the probability that at one (or two) of your horses came in first or second.

26

Example (from example 4.8)

An urn contains ten balls labeled 1..10.

Two balls are drawn one together.

- What is the sample space?
- What is the probability that the sum is odd?

Two balls are drawn one after the other without replacement.

- What is the sample space?
- What is the probability that the sum is odd?

Two balls are drawn one after the other with replacement

- What is the sample space?
- What is the probability that the sum is odd?

27

Example (from CL example 29)

Six light bulbs are chosen at random from 18 light bulbs of which 8 are defective. What is the probability that exactly two of the chosen bulbs are defective?

- B = total number of balls
- D = total number of defective balls
- b = number of balls chosen
- Probability space = set of ways to choose b bulbs from B with uniform distribution
- $E(B, D, b, d)$ = event with all selections of b from B where D bulbs are defective and d of the selected bulbs are defective
- Want $P(E(B, D, b, d))$ (called the *hypergeometric probability distribution*)

28