

CSE 21—Mathematics for Algorithm and System Analysis

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Day 2
Basic Enumeration and Counting

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Arrangements

How many arrangements of the letters 'l', 'a', 'j', 'o', 'l', 'l', 'a'?

- We have 3 l's to place
 - Choose 3 places from 7 total
 - We have 2 a's to place
 - Choose 2 places from 4 remaining
 - We have one 'j' to place
 - Choose 1 place from 2 remaining
 - We have one 'o' to place
 - Choose 1 place from 1 remaining
- $C(7, 3) \cdot C(4, 2) \cdot C(2, 1) \cdot C(1, 1)$

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Arrangements (multinomial coefficients)

If there are n objects with r_1 of type 1, r_2 of type 2, ..., r_m of type m , where $r_1 + r_2 + \dots + r_m = n$, then the number of arrangements of these n objects is:

$$\begin{aligned}
 P(n; r_1, r_2, \dots, r_m) &= \binom{n}{r_1, r_2, \dots, r_m} \\
 &= \binom{n}{r_1} \binom{n-r_1}{r_2} \binom{n-r_1-r_2}{r_3} \dots \binom{n-r_1-r_2-\dots-r_{m-1}}{r_m} \\
 &= \frac{n!}{r_1! r_2! \dots r_m!}
 \end{aligned}$$

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Arrangements

How many ways to select six ice cream cones from three flavors of ice cream?

- If the three flavors of ice cream are vanilla (V), chocolate (C), and Strawberry (S), then we can look at an order as # V followed by # C followed by #S. Imagine writing down the order in three columns:

- V		C		S
xx		xxx		x
- How many way to create an arrangement of six 'X's, and two 'l's'?
- $P(8; 6, 2) = 8! / (6! 2!) = 28$

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Selection with repetition

The number of selections with repetition of r objects from n types is $C(r+n-1, r)$

- Number of ways to create an arrangement of r 'X's and n 'I's is:
- $P(r+n-1; r, n-1) = (r+n-1)! / (r!(n-1)!) = C(r+n-1, r)$

Equivalent to the number of ways to distribute r identical objects into n distinct boxes

Equivalent to the number of non-negative integer solutions to $x_1 + x_2 + \dots + x_n = r$

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Arrangements

How many ways to select nine ice cream cones from three flavors of ice cream if we have to have at least one of each variety?

- Get one of each flavor
- Now, select six ice cream cones from three different flavors
 - If the three types of ice cream are vanilla (V), chocolate (C), and Strawberry (S), then we can look at an order as # V followed by # C followed by #S. Imagine writing down the order in three columns:

- V		C		S
XX		XXX		X
 - How many way to create an arrangement of six 'X's, and two 'I's?
 - $P(8; 6, 2) = 8! / (6!2!) = 28$

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Example (from 5.3 example 3)

Nine students (three from class A, three from class B and three from class C) bought a block of nine seats for homecoming game. What is the probability that the three A students, three B students and three C students will each be seated consecutively?

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Example (from 5.3 example 4)

DNA string consists of A, C, G, T.

Suppose a C-enzyme (breaks up DNA after each appearance of C) breaks a 20-letter DNA string into 8 fragments: AC, AC, AAATC, C, C, C, TATA, TGGC (all fragments except last must end in C). How many different strings could have caused these fragments?

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Example (from 5.3 example 5)

How many ways are there to form a sequence of 10 letters from 4 **as**, 4 **bs**, 4 **cs**, and 4 **ds** if each letter must appear at least twice?

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Example (from 5.3 example 6)

How many ways are there to fill a box of a dozen doughnuts chosen from 5 varieties if there must be at least one doughnut of each variety.

If there must be at least two of each variety?

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Example (from 5.3 example 7)

How many ways to pick a collection of exactly 10 balls from a pile of red balls, blue balls, and purple balls if there must be at least 5 red balls?

If at most 5 red balls?

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Example (from 5.3 Example 8)

How many arrangements of **b, a, n, a, n, a** such that:

- The **b** is immediately followed by an **a**?
- The pattern **bnn** never occurs.
- The **b** occurs before any of the **as**

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Example (from 5.3 exercise 22)

How many ways are there first to pick a subset of r people from 50 people (each of a different height) and next to pick a second subset of s people such that everyone in the first subset is shorter than everyone in the second subset?

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Distributions

Distributions of distinct objects

- Same as arrangements
- Distributing r distinct objects into n different boxes:
 - Put the r objects into a row
 - Stamp out n box names on each object

Distributions of identical objects

- Same as selections
- Distributing r identical objects into n different boxes
 - Choose a subset of r box names with repetition from n boxes
 - $C(r+n-1, r)$ distributions

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Example (from 5.4 example 1)

How many ways are there to assign 100 different diplomats to five different countries?

If 20 diplomats must be assigned to each country?

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Example

You're playing a card game and your two opponents have 4 spades between them. Which is more likely: that they're split 3-1, or 2-2?

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Example (from 5.4 Example 2)

In a game of bridge (N, S, E, W each dealt 13 cards). What is the probability that West has all 13 Spades?

What is the probability that each hand has one Ace?

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Example (from 5.4 example 4)

How many ways to distribute four identical oranges and six distinct apples into five distinct boxes?

In what fraction of these distributions does each box get exactly two objects?

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Example (from 5.4 example 5)

Show the number of ways to distribute r identical balls into n distinct boxes with at least one ball in each box is $C(r-1, n-1)$.

With at least r_1 balls in the first box, ..., r_n balls in the n th box, the number is $C(r-r_1-r_2-\dots-r_n+n-1, n-1)$

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Example 6 (from 5.4 example 6)

How many integer solutions are there to the equation $x_1+x_2+x_3+x_4=12$ ($x_i \geq 0$)?

With $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?

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Example (similar to 5.4 example 8)

A bitonic sequence is a sequence of 1s followed by 0s followed by 1s, or a sequence of 0s followed by 1s followed by 0s. 010, 011110, 10 are bitonic sequences. How many bitonic sequences of length 9 are there?

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Example (from 5.4 Exercise 20)

What fraction of all arrangements of EFFLORESCENCE has consecutive **C**s and consecutive **F**s but no consecutive **E**s?

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All of These are Equivalent

1. The number of ways to select r objects with repetition from n different types of objects
2. The number of ways to distribute r identical objects into n distinct boxes
3. The number of nonnegative integer solutions to:
 - $x_1 + x_2 + \dots + x_n = r$.

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Summary

Arrange/Select/Distribute r objects from n items

	Arrangement (ordered outcome) or Distribution of distinct objects	Combination (unordered outcome) or Distribution of identical objects
No repetition		
Unlimited repetition		
Restricted repetition		

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Binomial Theorem

Let's look at $(x+y)^n$

- $(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

- How many terms of the form x^ky^{3-k} in $(x+y)^3$?

How many ways to choose k **x**s and $(3-k)$ **y**s from 3 total?

- $C(3, k)$

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Binomial Theorem

Let's look at $(x+y)^n$

- What is the coefficient of x^ky^{n-k} in $(x+y)^n$?

$$\binom{n}{k}$$

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{k}x^{n-k}y^k + \dots \binom{n}{n}y^n$$

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Binomial Identities

Symmetry Identity

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Fundamental Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Algebraic Proof
- Proof by Pascal's triangle
- Proof by combinatorial argument

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Pascal's Triangle

Number of paths equals $C(n, k)$

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