

CSE 21—Mathematics for Algorithm and System Analysis

Summer, 2005

Day 1

Introduction

Basic Enumeration and Counting

Instructor: Neil Rhodes

Introduction

What will you be learning?

- Systematic analysis of different possibilities
- Exploration of the logical structure of a problem
- Ingenuity

Why?

- Combinatorial reasoning necessary for analyzing computer systems
 - Speed
 - Structure
- Optimization
- Measurement
- Probability

What is the structure?

- Lots and lots of word problems

Overview

Basic Enumeration and Counting

Inclusion/Exclusion

Elementary applied discrete probability

Functions

Decision Trees

Recurrence Relations

Generating Functions

Graph Theory

Asymptotic Notation

Administrative Details

Homework

- Assigned every class
- Need not be turned in

Reading

- Two textbooks
 - Tucker
 - Bender
- Read readings *before* class
- Print slides *before* class (available sometime the previous day)
- Take notes on slides *during* class
- Read slides (including notes) *after* class
- Read readings again, *after* class

Quizzes

- From first day to last day, exclusive:)
- Taken *directly* from homework

Midterms

- Two: Day 4 and Day 8

Final

- Day 10

Practice, practice, practice!

- 5 weeks is not much. Spend lots of time practicing.

Example Problem 1

There are 10 CSE students, 20 Math students, and 15 Psychology students. How many ways are there to choose 2 students from a different department?

Example Problem 2

How many different ways to create a 3-letter sequence using only a, b, c, d, e, f?

- With repetition
- Without repetition
- Without repetition and containing at least one E
- With repetition and containing at least one E

Example Problem 3

If a 5-card hand is chosen at random, what is the probability of a flush (all 5 cards of the same suit)?

Example Problem 4

How many ways are there to distribute 25 different presents to four people (including the boss) at an office party so that the boss receives exactly twice as many presents as the second most popular person?

Collections of Objects

Types of collections

- Set
 - Standard notation: $\{\text{element}_1, \text{element}_2, \dots, \text{element}_n\}$
 - Duplicates not allowed
 - Order doesn't matter
- List
 - Notation: $(\text{element}_1, \text{element}_2, \dots, \text{element}_n)$
 - Duplicates are allowed
 - Order **does** matter
- Multiset (bag)
 - Like set, but duplicates **are** allowed
 - Not often used

Sizes

- k -list: list with k elements (similarly, k -set and k -multiset)
- $|A|$: cardinality of A
 - number of elements in A

Counting

Given the set $S = \{x, y, z\}$

How many ways to list (without duplicates) elements of S ?

- Permutations of S

How many ways to list (with duplicates) elements of S ?

How many ways to construct a k -list of distinct elements from S ?

How many ways to construct a 2-set of (distinct) elements?

How many ways to construct a 2-multiset of elements?

Rule of Product

If we're looking at outcomes by making a sequence of k choices such that:

- The i th choice can be made in c_i ways
 - c_i independent of previous choices
- Each outcome can occur in only one way
 - outcomes are all distinct

then the number of outcomes is $c_1 \cdot c_2 \cdot \dots \cdot c_n$

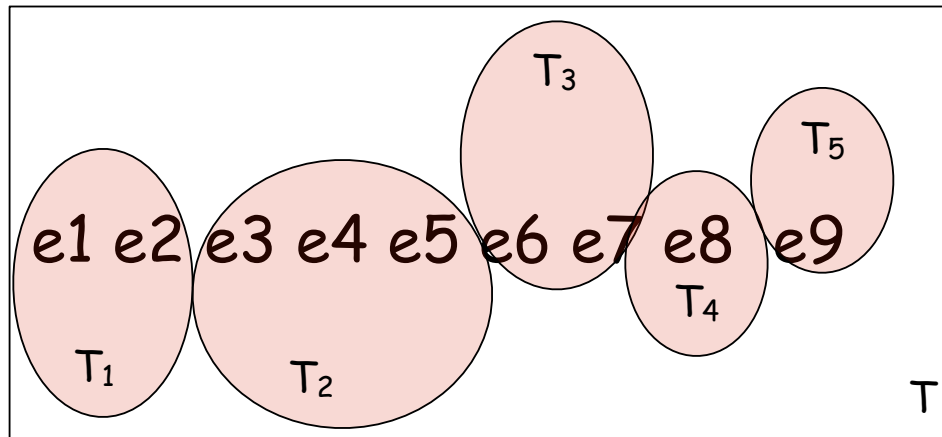
Example of Rule of Product

There are five Prom King candidates and six Prom Queen candidates. How many ways are there to pick a pair of Prom King and Queen?

Rule of Sum

If a set of T elements can be partitioned into sets $T_1, \dots, T_j$, such that each element in T appears in exactly one set T_i , then:

- $|T| = |T_1| + |T_2| + \dots + |T_j|$



Example of Rule of Sum (similar to 5.1 Example 1)

There are 100 Math majors and 90 CS majors. How many total students?

- No double-majors
- 5 double-majors

Example of both Rules (similar to CL example 5)

There are 10 CSE students, 20 Math students, and 15 Psychology students. How many ways are there to choose 2 students from a different department

- Rule of Sum
 - CSE/Math

 - CSE/Psychology

 - Psychology/Math

Lexicographic Ordering

“Dictionary” ordering

- Words
 - $A < AA < AAA < B < BAT < \dots < Z < ZY < ZZ < ZZZ < ZZZZZZZZ$
- Numbers (add leading zeros so that each number has the same number of digits)
 - $000 < 009 < 010 < 157 < 158 < 999$
- Lists (elements within the list need a defined ordering)
 - $red < green < blue$
 - $A < AA < AAA$
 - $05 < 10 < 12$
 - $(red, A, 05) < (red, A, 10) < (red, A, 12) < (red, AA, 05) < (red, AA, 10) < (red, AA, 12) < (red, AAA, 05) < (red, AAA, 10) < (red, AAA, 12) < \dots < (blue, AAA, 12)$

Example (from CL Example 4)

We form nonsense words using the following rules:

- Letters are from: A, I, L, S, T.
- Each word has 4 letters: 2 vowels, 2 consonants
- No adjacent vowels
- If consonants are adjacent, they are different

What is the first word in lexicograph order?

The last word?

How many total words are possible?

Counting

Come up with a way to count so that:

- Everything gets counted once
- Nothing gets counted twice

Try to break up a problem into manageable subproblems

Find a structure in the problem that allows it to be broken into subproblems or stages

And/Or

Break up problem using AND (AND THEN) and OR

- AND should become product
- OR should become sum

Examples (from 5.1 example 4)

How many different ways to create a 3-letter sequence using only a, b, c, d, e, f?

- With repetition
- Without repetition
- Without repetition and containing at least one E
- With repetition and containing at least one E

Permutations and Combinations

Permutation of n objects: ordering of all n objects

r -permutation: ordering of r of n objects

- Order matters
- $P(n, 2) = n(n-1)$
 $P(n, 3) = n(n-1)(n-2)$
 $P(n, r) = n(n-1)(n-2)\dots(r-1)$
 $P(n, n) = n(n-1)\dots(2)(1) = n!$
- $P(n, r) = n(n-1)(n-2)\dots(r-1) = n!/(n-r)!$

r -combination of n objects: choosing r of n objects (unordered)

- $C(n, r) = \# \text{ of } r\text{-permutations} / \# \text{ of ways to order } r \text{ items}$
 $= P(n, r) / P(r, r)$
 $= n!/(n-r)! / r!$
 $= n!/(n-r)!r!$
- $C(n, r) = C(n, n-r)$

$$\binom{n}{r} = C(n, r) = \frac{n(n-1)\dots(n-r-1)}{r!} = \frac{n!}{r!(n-r)!}$$

Example Problem (from CL example 1)

How many ways are there to rank n candidates for the job of chief wizard?

If rankings are chosen at random, what is the probability that the fifth candidate is in second place?

Example Problem (from 5.2 example 4)

If a 5-card hand is chosen at random, what is the probability of a flush (all 5 cards of the same suit)?

- How many 5-card hands are there?
- How many 5-card hands are flushes?

Example Problem (from 5.2 example 2)

How many ways are there to arrange the seven letters in the word **SYSTEMS**?

In how many do the three **S** letters appear next to each other?

In how many does the **E** occur somewhere before the **M**?

In how many do the three S letters appear next to each other *and* the **E** occurs somewhere before the **M**?

Example Problem (from 5.2 exercise 40)

What is the probability that a five-card hand has at least one card of each suit?

What is the probability that a six-card hand has at least one card of each suit?

