University of California, San Diego
Department of Computer Science and Engineering
Course: CSE 105 − Theory of Computability

Instructor: Adriana Palacio
2004 Summer Session I
July 29, 2004

Test 2 Solutions with Explanations

Problem 1 [**20 points**]

Indicate whether the following statements are TRUE (T) or FALSE (F) by **circling** your answers. **You will receive 2 points for each correct answer, 0 points for each missing answer, and -1 point for each incorrect answer.**

a) A language and its complement can both be infinite.

TRUE; Let $A = \{ w \in \{\mathtt{0},\mathtt{1}\}^* \mid w \text{ starts with a } \mathtt{1} \}$. Then A and its complement $\overline{A} = \{ w \in \{\mathtt{0},\mathtt{1}\}^* \mid w \text{ starts with a } \mathtt{0} \}$ are both infinite.

b) For any languages A and B such that $A \subseteq B$, if B is context free then A is context free.

FALSE; Let $A = \{ \mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \in \mathbb{N} \}$ and $B = \{\mathtt{a},\mathtt{b},\mathtt{c}\}^*$. Then B is context free and $A \subseteq B$, but A is not context free.

This question is very similar to question 1. b) from Test 1.

c) For any languages A and B such that $A \subseteq B$, if A is context free then B is context free.

FALSE; Let $A = \emptyset$ and $B = \{ \mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \in \mathbb{N} \}$. Then A is context free and $A \subseteq B$, but B is not context free.

This question is analogous to the previous one.

d) If a language is context free, then its complement is decidable.

TRUE; If a language L is context free, then it is decidable. Since the class of decidable languages is closed under complementation, $\overline{L}$ is decidable.

e) If a language is recognizable, then it is decidable.

FALSE; There exist recognizable languages that are not decidable. We will see examples of these this week.

f) If a language is not decidable, then its complement is not decidable.

TRUE; Proof by contradiction. Let L be a language that is not decidable. Assume that its complement $\overline{L}$ is decidable. Since the class of decidable languages is closed under complementation, the complement of $\overline{L}$ is decidable. But the complement of $\overline{L}$ is L, so L is decidable. This contradicts the initial choice of L as a language that is not decidable.

g) A Turing machine may have infinitely many states.

FALSE; This follows from the formal definition of a TM. The set of states of a TM is finite.

h) If the complement of a language is finite then the language is context free.

TRUE; Let L be a language such that $\overline{L}$ is finite. Since all finite languages are regular, $\overline{L}$ is regular. Since the class of regular languages is closed under complementation, $\overline{\overline{L}} = L$ is regular. Since all regular languages are context free, L is context free.

i) A class of languages may be closed under union and complement, and not be closed under intersection.

FALSE; Let $\mathcal{C}$ be any class of languages that is closed under union and complementation. Let A and B be arbitrary languages in $\mathcal{C}$. Since $\mathcal{C}$ is closed under complementation, $\overline{A} \in \mathcal{C}$ and $\overline{B} \in \mathcal{C}$. Since $\mathcal{C}$ is closed under union, $\overline{A} \cup \overline{B} \in \mathcal{C}$. Since $\mathcal{C}$ is closed under complementation, $\overline{\overline{A} \cup \overline{B}} \in \mathcal{C}$. By De Morgan's law, $A \cap B = \overline{\overline{A} \cup \overline{B}} \in \mathcal{C}$. This proves that $\mathcal{C}$ is closed under intersection. Thus, any class of languages that is closed under union and complementation is closed under intersection.

j) If a class of languages is closed under intersection but not closed under union, then we can conclude that it is not closed under complement.

TRUE; Proof by contradiction. Let $\mathcal{C}$ be a class of languages that is closed under intersection, but not closed under union. Assume that $\mathcal{C}$ is closed under complement. With an argument similar to that used in part **i)**, we can prove that $\mathcal{C}$ is closed under union. This contradicts the initial choice of L.

Problem 2 [**20 points**] Let

$$A \quad = \quad \{\, a^m b^{m+n} c^n \mid m, n \geq 0 \,\}.$$

Specify a CFG G such that $L(G) = A$ and G has at most 7 rules.

First, we observe that if $A_1 = \{\, a^m b^m \mid m \geq 0 \,\}$ and $A_2 = \{\, b^n c^n \mid n \geq 0 \,\}$, then $A = A_1 \cdot A_2$.

The following are rules for grammars generating A_1 and A_2, respectively.

$S_1 \rightarrow aS_1b \mid \varepsilon$

$S_2 \rightarrow bS_2c \mid \varepsilon$

We obtain a grammar G such that $L(G) = A$ using the construction we used to prove that the class of CFLs is closed under concatenation. Namely, let $G = (V, \Sigma, R, S)$, where

- $V = \{S_1, S_2, S\}$
- $\Sigma = \{a, b, c\}$
- $R = \{S \rightarrow S_1S_2,\ S_1 \rightarrow aS_1b \mid \varepsilon,\ S_2 \rightarrow bS_2c \mid \varepsilon\}$
- S is the start variable

Grammar G has 5 rules.

Problem 3 [**20 points**] Let

$$A \;=\; \{\, a^m b^n \mid n = m^2 \,\}.$$

a) Prove that A is not context free.

Proof by contradiction. Assume that A is a CFL. Then the Pumping Lemma for CFLs applies to A. Let p be the pumping length given by the P.L. Let $s = a^p b^{p^2}$. Then $s \in A$ and $|s| \geq p$. By the P.L., s can be split into $uvxyz$ such that the following conditions hold:

(1) $|vy| > 0$

(2) $|vxy| \leq p$

(3) For all $i \geq 0$: $uv^i xy^i z \in A$

Comment: *Recall that we can choose s, but we have no control over u, v, x, y, z. We know, however, that they satisfy the three conditions. Now our goal is to show that conditions* (1) *and* (2) *imply that for some $i \geq 0$: $uv^i xy^i z$ is not in A, contradicting condition* (3) *above. This means our assumption that A was a CFL is false.*

We show that $uv^2xy^2z \notin A$.

If uv^2xy^2z has only p a's then, by condition (1), uv^2xy^2z has more than p^2 b's. Therefore, the number of b's in this string is not the square of the number of a's (the number of a's is p and the number of b's is greater than p^2). Hence $uv^2xy^2z \notin A$.

If uv^2xy^2z has at least $p+1$ a's, then it would have to have at least $(p+1)^2 = p^2 + 2p + 1$ b's in order for it to be in A (i.e., vy would have to include $2p+1$ additional b's). Since $2p+1 > p$ and $|vy| \leq |vxy| \leq p$ (by condition (2)), it is impossible for uv^2xy^2z to have enough b's. Therefore, $uv^2xy^2z \notin A$.

b) Prove that A is not regular.

Proof by contradiction. Assume that A is regular. Since all regular languages are context free, A is context free. This contradicts part **a)**.

Comment: *This can also be proved using the Pumping Lemma for regular languages, but that would take* **much more time**. *This is an example of how taking the time to think about a problem before starting to write up a solution can pay off.*

c) Prove that the following language is decidable. (You may use the fact that A is not context free, even if you did not prove part **a)**.)

$$B \;=\; \{\, \langle M \rangle \mid M \text{ is a PDA and } L(M) = A \,\}.$$

Since A is not context free, there are no PDAs M such that $L(M) = A$. Therefore, $B = \emptyset$. This language is regular. Since all regular language are decidable, B is decidable.

Comment: *The first things to do here is understand what language B is. Once you figure that out, the problem becomes very simple.*

Another way to show that $\emptyset$ is decidable is to contruct a TM that decides it. A TM that rejects on all inputs is a decider for this language.

Problem 4 [**20 points**] Let $\Sigma = \{\mathtt{0}, \mathtt{1}\}$. Solve **ONE** of the following two problems.

A) Prove that the following language is decidable:

$$A \;=\; \{\, \langle M \rangle \mid M \text{ is a DFA and } L(M) = 0^* \,\}.$$

Let D be a DFA that recognizes 0^*. Let T be a TM that decides the language EQ_{DFA}. The following TM, which has the descriptions of D and T hardwired, decides A.

$M_A =$ On input $\langle M \rangle$
Run $T(\langle M, D \rangle)$
If it accepts then *accept* else *reject*

B) Prove that the following language is decidable:

$$B \;=\; \{\, \langle M \rangle \mid M \text{ is a DFA that accepts some string beginning with 0} \,\}.$$

Let D be a DFA that recognizes the language $\{\, w \in \{\mathtt{0}, \mathtt{1}\}^* \mid w \text{ begins with } \mathtt{0} \,\}$. Let T be a TM that decides the language E_{DFA}.

Let U be a TM that on input $\langle D_1, D_2 \rangle$, where D_1 and D_2 are DFAs, returns $\langle N \rangle$, where N is a DFA such that $L(N) = L(D_1) \cap L(D_2)$. Notice that this TM exists since the transformation underlying the proof that the class of regular languages is closed under intersection can be automated.

The following TM, which has the descriptions of D, T and U hardwired, decides B.

$M_B =$ On input $\langle M \rangle$
Let $\langle N \rangle$ be the output of $U(\langle M, D \rangle)$
Run $T(\langle N \rangle)$
If it accepts then *reject* else *accept*

Comment: *There are many other ways to solve this problem. Some students came up with some pretty creative solutions.*

Problem 5 [**20 points**]

a) Consider the operation "$-$" defined as follows, for languages A and B:

$$A - B = \{ w \mid w \in A \text{ and } w \notin B \}$$

Prove that the class of decidable languages is closed under $-$.

We observe that for any two languages A and B, $A - B = A \cap \overline{B}$.

Let A and B be decidable languages. Since the class of decidable languages is closed under complementation and intersection, $A \cap \overline{B}$ is decidable. Therefore, $A - B$ is decidable.

Comment: *It is also possible to construct a decider for $A - B$, using deciders for A and B.*

b) Consider the operation "@" defined as follows, for languages A and B:

$$A \,@\, B = \{ w \mid w \in A \text{ and there exists } x \in B \text{ such that } |x| = |w| \}$$

Prove that the class of recognizable languages is closed under @.

Let A and B be recognizable languages over an alphabet Σ. Let M_A and M_B be TMs that recognize A and B, respectively. The following TM recognizes $A \,@\, B$.

$M =$ On input a string w
    Run $M_A(w)$
    If it accepts then
        Run $M_B(x)$ for every $x \in \Sigma^*$ such that $|x| = |w|$ in parallel
        If any of these computations accepts then *accept* EndIf
    EndIf
    Reject