University of California, San Diego
Department of Computer Science and Engineering
Course: CSE 105 − Theory of Computability

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Test 2 Solutions

For some of the problems, there are many possible (correct) solutions. I present one for each.

Problem 1 [**20 points**] Indicate whether the following statements are TRUE or FALSE.

a) A language and its complement can both be infinite.
TRUE

b) For any languages A and B such that $A \subseteq B$, if B is context free then A is context free.
FALSE; **This question is very similar to question 1. b) from Test 1.**

c) For any languages A and B such that $A \subseteq B$, if A is context free then B is context free.
FALSE; **This question is analogous to the previous one.**

d) If a language is context free, then its complement is decidable.
TRUE

e) If a language is recognizable, then it is decidable.
FALSE

f) If a language is not decidable, then its complement is not decidable.
TRUE

g) A Turing machine may have infinitely many states.
FALSE

h) If the complement of a language is finite then the language is context free.
TRUE

i) A class of languages may be closed under union and complement, and not be closed under intersection.
FALSE

j) If a class of languages is closed under intersection but not closed under union, then we can conclude that it is not closed under complement.
TRUE

Problem 2 [**20 points**] Let

$$A \quad = \quad \{\, a^m b^{m+n} c^n \mid m, n \geq 0 \,\}.$$

Specify a CFG G such that $L(G) = A$ and G has at most 7 rules.

Let $G = (V, \Sigma, R, S)$, where

- $V = \{S_1, S_2, S\}$
- $\Sigma = \{a, b, c\}$
- $R = \{S \ \rightarrow \ S_1 S_2, \ \ S_1 \ \rightarrow \ aS_1 b \mid \varepsilon, \ \ S_2 \ \rightarrow \ bS_2 c \mid \varepsilon\}$
- S is the start variable

Problem 3 [**20 points**] Let

$$A \quad = \quad \{\, a^m b^n \mid n = m^2 \,\}.$$

a) Prove that A is not context free.

Proof by contradiction. Assume that A is a CFL. Let p be the pumping length given by the P.L. for CFLs. Let $s = a^p b^{p^2}$. Then $s \in A$ and $|s| \geq p$. By the P.L., s can be split into $uvxyz$ such that the conditions of the P.L. hold. We show that $uv^2xy^2z \notin A$.

If uv^2xy^2z has only p a's then, since $|vy| > 0$, uv^2xy^2z has more than p^2 b's. Therefore, the number of b's in this string is not the square of the number of a's (the number of a's is p and the number of b's is greater than p^2). Hence $uv^2xy^2z \notin A$.

If uv^2xy^2z has at least $p+1$ a's, then it would have to have at least $(p+1)^2 = p^2 + 2p + 1$ b's in order for it to be in A (i.e., vy would have to include $2p+1$ additional b's). Since $2p+1 > p$ and $|vy| \leq |vxy| \leq p$, it is impossible for uv^2xy^2z to have enough b's. Therefore, $uv^2xy^2z \notin A$.

b) Prove that A is not regular.

Proof by contradiction. Assume that A is regular. Since all regular languages are context free, A is context free. This contradicts part **a)**.

c) Prove that the following language is decidable. (You may use the fact that A is not context free, even if you did not prove part **a)**.)

$$B \quad = \quad \{\, \langle M \rangle \mid M \text{ is a PDA and } L(M) = A \,\}.$$

Since A is not context free, there are no PDAs M such that $L(M) = A$. Therefore, $B = \emptyset$. This language is regular. Since all regular languages are decidable, B is decidable.

Problem 4 [**20 points**] Let $\Sigma = \{0, 1\}$. Solve **ONE** of the following two problems.

A) Prove that the following language is decidable:

$$A \;=\; \{\, \langle M \rangle \mid M \text{ is a DFA and } L(M) = 0^* \,\}.$$

Let D be a DFA that recognizes 0^*. Let T be a TM that decides the language EQ_{DFA}. The following TM, which has the descriptions of D and T hardwired, decides A.

$M_A =$ On input $\langle M \rangle$
Run $T(\langle M, D \rangle)$
If it accepts then *accept* else *reject*

B) Prove that the following language is decidable:

$$B \;=\; \{\, \langle M \rangle \mid M \text{ is a DFA that accepts some string beginning with } 0 \,\}.$$

Let D be a DFA that recognizes the language $\{\, w \in \{0, 1\}^* \mid w \text{ begins with } 0 \,\}$. Let T be a TM that decides the language E_{DFA}.

Let U be a TM that on input $\langle D_1, D_2 \rangle$, where D_1 and D_2 are DFAs, returns $\langle N \rangle$, where N is a DFA such that $L(N) = L(D_1) \cap L(D_2)$. Notice that this TM exists since the transformation underlying the proof that the class of regular languages is closed under intersection can be automated.

The following TM, which has the descriptions of D, T and U hardwired, decides B.

$M_B =$ On input $\langle M \rangle$
Let $\langle N \rangle$ be the output of $U(\langle M, D \rangle)$
Run $T(\langle N \rangle)$
If it accepts then *reject* else *accept*

Problem 5 [**20 points**]

a) Consider the operation "$-$" defined as follows, for languages A and B:

$$A \;-\; B \;=\; \{\, w \mid w \in A \text{ and } w \notin B \,\}$$

Prove that the class of decidable languages is closed under $-$.

We observe that for any two languages A and B, $A - B = A \cap \overline{B}$.

Let A and B be decidable languages. Since the class of decidable languages is closed under complementation and intersection, $A \cap \overline{B}$ is decidable. Therefore, $A - B$ is decidable.

b) Consider the operation "@" defined as follows, for languages A and B:

$$A \mathbin{@} B \;=\; \{\, w \mid w \in A \text{ and there exists } x \in B \text{ such that } |x| = |w| \,\}$$

Prove that the class of recognizable languages is closed under @.

Let A and B be recognizable languages over an alphabet Σ. Let M_A and M_B be TMs that recognize A and B, respectively. The following TM recognizes $A \mathbin{@} B$.

$M =$ On input a string w
Run $M_A(w)$
If it accepts then
Run $M_B(x)$ for every $x \in \Sigma^*$ such that $|x| = |w|$ in parallel
If any of these computations accepts then *accept* EndIf
EndIf
Reject