Lecture 4 Overview

- Regular operations on languages
- Closure properties of the class of regular languages
- Regular expressions
- Equivalence of regular expressions and finite automata

Regular Operations on Languages

Def: Let L, L_1 and L_2 be languages over alphabets Σ, Σ_1, and Σ_2, respectively. The regular operations **union**, **concatenation**, and **star** are defined as follows.

Union: $L_1 \cup L_2 = \{w \in {\Sigma_1}^* \cup {\Sigma_2}^* \mid w \in L_1 \text{ or } w \in L_2\}$

Concatenation: $L_1 \cdot L_2 = L_1 L_2 =$
$\{w \in (\Sigma_1 \cup \Sigma_2)^* \mid w = w_1 w_2 \text{ for some } w_1 \in L_1, w_2 \in L_2\}$

Star: $L^* =$
$\{w \in \Sigma^* \mid w = w_1 \cdots w_n \text{ for some } n \geq 0, w_1, \ldots, w_n \in L\}$

Closure Properties of the Class of Regular Languages

Theorem: If L, L_1, and L_2 are regular languages over an alphabet Σ, then the following languages are also regular.

1) $\overline{L} = \{w \in \Sigma^* \mid w \notin L\}$
2) $L_1 \cup L_2 = \{w \in \Sigma^* \mid w \in L_1 \text{ or } w \in L_2\}$
3) $L_1 \cap L_2 = \{w \in \Sigma^* \mid w \in L_1 \text{ and } w \in L_2\}$
4) $L_1 \cdot L_2 = \{w \in \Sigma^* \mid w = w_1 w_2 \text{ for } w_1 \in L_1, w_2 \in L_2\}$
5) $L^* =$
   $\{w \in \Sigma^* \mid w = w_1 \cdots w_n \text{ for } n \geq 0, w_1, \ldots, w_n \in L\}$

Before proving this, let's consider some examples.

Closure Properties – Examples

Let $\Sigma = \{0, 1\}$.

Ex. 1)

$L = \{w \in \{0,1\}^* \mid w \text{ does not contain substring } 1010\}$

Is L regular?

Closure Properties – Examples

Ex. 2) Prove that the following language is regular.
$L = \{w \mid w$ starts with 110 and contains substring 1010$\}$

Closure Properties – Proof of Theorem

Recall the theorem we proved yesterday:

Theorem: A language L is regular if and only if there exists an NFA that recognizes L.

We will use this result.

Closure Properties – Proof of Theorem

Proof of Theorem – part 1):

Given: L is regular, i.e., there exists a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes L.

Want: $\overline{L} = \{w \in \Sigma^* \mid w \notin L\}$ is regular, i.e., there exists an NFA N that recognizes $\overline{L}$. We will build N. (We could build a DFA instead, but NFAs are often simpler.)

Construction: $N = (Q', \Sigma, \delta', q_0', F')$, where

$Q' =$ $\delta' =$ $q_0' =$ $F' =$

Closure Properties – Proof of Theorem

Proof of Theorem – part 2):

Given: L_1 and L_2 are regular, i.e., there exist DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ that recognize L_1 and L_2, respectively.

Want: $L_1 \cup L_2 = \{w \in \Sigma^* \mid w \in L_1 \text{ or } w \in L_2\}$ is regular, i.e., there exists an NFA N that recognizes $L_1 \cup L_2$. We will build N.

Construction: $N = (Q, \Sigma, \delta, q_0, F)$, where

$Q =$ $\delta =$ $q_0 =$ $F =$

Closure Properties – Proof of Theorem

Nondeterminism makes the construction easy!

Construction: $N = (Q, \Sigma, \delta, q_0, F)$, where

Closure Properties – Proof of Theorem

Proof of Theorem – part 3):

Given: L_1 and L_2 are regular, i.e., there exist DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ that recognize L_1 and L_2, respectively.

Want: $L_1 \cap L_2 = \{w \in \Sigma^* \mid w \in L_1 \text{ and } w \in L_2\}$ is regular, i.e., there exists an NFA N that recognizes $L_1 \cap L_2$.

We could build N, but there is an easier way to prove that $L_1 \cap L_2$ is regular.

Closure Properties – Proof of Theorem

Given: L_1 and L_2 are regular.

Want: $L_1 \cap L_2$ is regular.

Closure Properties – Proof of Theorem

Proof of Theorem – part 4):

Given: L_1 and L_2 are regular, i.e., there exist DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ that recognize L_1 and L_2, respectively.

Want:
$L_1 \cdot L_2 = \{w \in \Sigma^* \,|\, w = w_1 w_2 \text{ for some } w_1 \in L_1, w_2 \in L_2\}$ is regular, i.e., there exists an NFA N that recognizes $L_1 \cdot L_2$. We will build N.

Closure Properties – Proof of Theorem

Again, nondeterminism makes the construction easy!

Construction: $N = (Q, \Sigma, \delta, q_0, F)$, where

Closure Properties – Examples

Ex. 3) Consider the following languages.

$L_1 = \{w \mid w \text{ starts with } 10\}$

$L_2 = \{w \mid w \text{ starts with } 111\}$

By part 2) of the theorem,

$L_1 L_2 =$

is regular.

Closure Properties – Examples

Ex. 4) Let $L = \{00, 01, 10, 11\}$. Then

$$L^* = \{w_1 \cdots w_n \mid n \geq 0 \text{ and } w_1, \ldots, w_n \in L\}$$
$$=$$

By part 5) of the theorem, L^* is regular.

Closure Properties – Proof of Theorem

Proof of Theorem – part 5):

Given: L is regular, i.e., there exists a DFA
$M = (Q, \Sigma, \delta, q_0, F)$ that recognizes L.

Want:
$L^* = \{w \in \Sigma^* \mid w = w_1 \cdots w_n \text{ for } n \geq 0, w_1, \ldots, w_n \in L\}$
is regular, i.e., there exists an NFA N that recognizes L^*.
We will build N.

Construction: $N = (Q', \Sigma, \delta', q_0', F')$, where

$Q' =$ $\delta' =$ $q_0' =$ $F' =$

Closure Properties – Proof of Theorem

Construction: $N = (Q', \Sigma, \delta', q_0', F')$, where

Regular Expressions