Lecture 3 Overview

- Things to remember
- Nondeterministic Finite Automaton (NFA): definition, examples
- Equivalence of DFAs and NFAs
- Introduction to regular operations and closure properties of regular languages

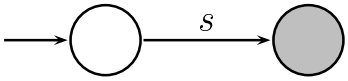
Things to remember

- $\emptyset$ is the empty **set**, i.e, the set that contains 0 elements: $\emptyset = \{\}$
- ε ("epsilon") is the empty **string**, i.e., the string that has 0 symbols (and hence length 0).
- We will not use $\emptyset$ or ε as alphabet symbols. These are **distinguished symbols** that have the meanings indicated above.
- $\{\varepsilon\}$ is not an alphabet! It is a language that consists of only the empty string.
- $\emptyset$ is a language too. The language that contains no strings at all.

Nondeterminism

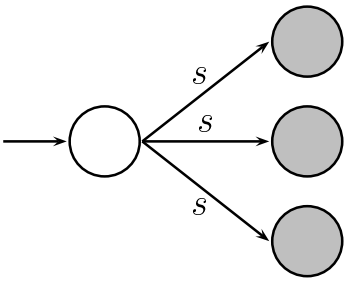
Nondeterminism allows several possible next states at every step.

Determinism:



one next state per s

Nondeterminism:



set of next states per s

State diagrams – NFA versus DFA

NFA

- a state may have 0, 1, or many exiting transitions for each symbol in the alphabet
- labels on the transition arrows are symbols from the alphabet or the distinguished symbol ε

DFA

- every state has exactly one exiting transition for each symbol in the alphabet
- labels on the transition arrows are symbols from the alphabet

Formal Definition of NFA

Def: A **nondeterministic finite automaton** (or **NFA**) is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of **states**
2. Σ is an alphabet
3. $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Q)$ is the **transition function**
4. $q_0 \in Q$ is the **start state**
5. $F \subseteq Q$ is the **set of accept (final) states**.

$\mathcal{P}(Q)$ is the set of all subsets of Q.

Computation for NFAs

An NFA M accepts a string $w \in \Sigma^*$ if there is some path to follow on w that ends in a final (i.e., accept) state.

M rejects w if all paths on w end in non-accepting states.

Formally, ...

Formal Definition of Computation for NFAs

Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA and w a string over Σ. M **accepts** w if there is a sequence of symbols $w_1, w_2, \ldots, w_n \in \Sigma \cup \{\varepsilon\}$, where $w = w_1 w_2 \cdots w_n$, and a sequence of states $r_0, r_1, \ldots, r_n \in Q$ such that

1. $r_0 = q_0$ (starts right)
2. $r_{i+1} \in \delta(r_i, w_{i+1})$ for $i = 0, \ldots, n-1$ (moves right)
3. $r_n \in F$ (ends right)

Formal Definition of Computation for NFAs

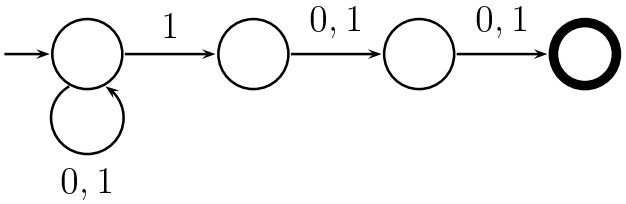
Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA. M **recognizes** a language L if $L = \{w \in \Sigma^* \mid M \text{ accepts } w\}$, i.e.,

For all $w \in \Sigma^*$:

- $w \in L \Rightarrow M$ accepts w
- $w \notin L \Rightarrow M$ rejects w

Notation: $L(M)$ denotes the language recognized by M.

NFA example (without ε-transitions)



ε-transitions

$$q_1 \xrightarrow{\varepsilon} q_2$$

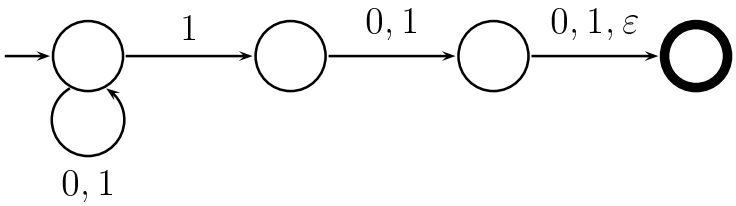
Without reading the next input symbol, the machine splits into two copies of itself.

One of them follows the ε-transition.

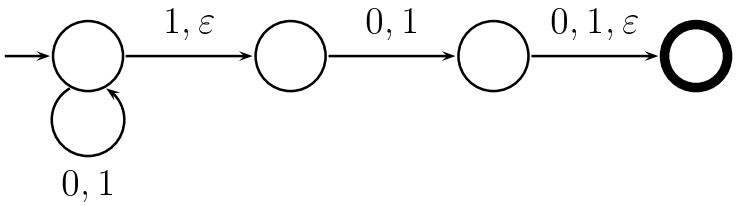
The other stays at q_1.

In parallel, each copy continues its nondeterministic computation.

NFA example (with ε-transitions)

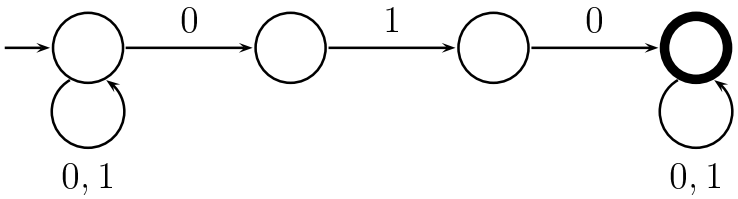


NFA example (with ε-transitions)



NFA example – formal description

M:



$M = (\{q_0, q_1, q_2, q_3\}, \{0, 1\}, \delta, q_0, \{q_3\})$, where

$$\begin{aligned} \delta(q_0, 0) &= \{q_0, q_1\} \\ \delta(q_0, 1) &= \{q_0\} \\ \delta(q_0, \varepsilon) &= \emptyset \\ &\vdots \end{aligned}$$

Why NFAs?

- NFAs seem more powerful than DFAs.
  We will show that they **are not**.
- Building NFAs is often easier than building DFAs
- NFAs may be more compact than DFAs that recognize the same language.

Is every DFA an NFA?

Equivalence of NFAs and DFAs

Two machines M_1, M_2 are **equivalent** if $L(M_1) = L(M_2)$.

DFAs and NFAs recognize the same class of languages: regular languages.

To show this, we will prove the following.

Equivalence of NFAs and DFAs

Theorem: A language L is regular if and only if there exists an NFA that recognizes L.

Proof: We must prove two statements:

1) If L is a regular language then there exists an NFA that recognizes it.

2) If there exists an NFA that recognizes a language L, then L is regular.

Equivalence of NFAs and DFAs

1) If L is a regular language then there exists an NFA that recognizes it.

Proof of 1):

Given: L is regular, i.e., there exists a DFA M that recognizes L.

Want: An NFA N that recognizes L.

Construction: Let $N = M$. Since any DFA is an NFA, N is an NFA that recognizes L.

Equivalence of NFAs and DFAs

2) If there exists an NFA that recognizes a language L, then L is regular.

Proof of 2):

Given: There exists an NFA N that recognizes L.

Want: L is regular, i.e., there exists a DFA M that recognizes L. We will build M.

Construction: