Section 4.1
Switching Algebra
Symmetric Functions

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Symmetric Functions

- A function in which each input variable plays the same role in determining the value of the function.
- Examples:
  - majority function: it is '1' only when more than half of the inputs are '1'. It is the "carry" function in the binary addition and the "voter" function used in fault tolerant computing;
  - (odd) parity function: is '1' only if an odd number of inputs are '1'. It is the "sum" function for binary addition and it is used in detecting or correcting code circuits.

Symmetric Functions

- Symmetric functions can be synthesized with fewer logic elements
- Detection of symmetry is an important and HARD problem in CAD
- There are several types of symmetry

Totally Symmetric Functions

Definition

- A function $f(x_1,x_2,........,x_n)$ is totally symmetric iff it is unchanged by any permutation of its variables.
- Examples:
  - $F = xy + xz + yz$ (majority function)
  - $F = x'y + xy'$ (parity function, exor)

Theorem:

- $f(x_1,x_2,........,x_n)$ is totally symmetric iff it can be specified by stating a list of integers $A=\{a_1,a_2,........,a_m\}$, $0=a_j=n$ so that $f=1$ iff exactly a_j of the variables are 1
- $\{a_1,a_2,........,a_m\}$ are called a-numbers
- S_A is the symbol used to indicate a Symmetric Function.

a-number Example

- Consider the function $S_1(x,y,z)$
- Then this function has a single a-number = 1
- The truth table is:

x	y	z	f
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

Another a-number Example

- Consider the function $S_{0,2}(x,y,z)$
- Then this function has two a-numbers, 0 & 2
- The truth table is:

x	y	z	f
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Problem

- **Write the truth table of $S_{1,2}(x, y, z)$**

x	y	z	f
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Mixed Symmetric Functions

Definition

- **A function $f(x_1,x_2,........,x_n)$ is mixed symmetric iff it is not totally symmetric, but it can be changed into totally symmetric by replacing some of its variables by their complements.**
- **A mixed symmetric function is represented by the symbol S_A (x,y',z') where y and z are the variables to be complemented**
- **Examples:**
  - ***F = xy'z'* is not totally symmetric, but *xyz* is totally symmetric, therefore F is mixed symmetric**

Partially Symmetric Functions

- A function $f(x_1,x_2,\ldots\ldots,x_n)$ will be called symmetric if it is either totally or mixed symmetric
- There are functions that are unchanged when some, but not all, of their variables are permuted. For example F=x+yz is unchanged by a permutation of y and z. Such functions are called partially symmetric.

Definition

- A function $f(x_1,x_2,\ldots\ldots,x_n)$ is partially symmetric in $x_{i1}, x_{i2}, \ldots, x_{im}$ iff it is unchanged by any permutation of the variables $x_{i1}, x_{i2}, \ldots, x_{im}$

Identification of symmetric functions

- Definition:
  column sum & row sum (a-number)

- Ex. f(x, y, z) = S (1, 2, 4, 7)

x	y	z	a-number	
0	0	1	1	← Row sum
0	1	0	1	← Row sum
1	0	0	1	← Row sum
1	1	1	3	← Row sum
2	2	2		

Column sum

Identification of symmetric functions

Naive Way:

- **If all variables are uncomplemented (complemented):**
  - **Expand to Canonical Form and count minterms for each possible a-number**
  - **If f = 1 for all minterms corresponding to an a-number, then that a-number is included in the set**

Question:

- **What is the total number of minterms for an a-number in a function of n variables?**

$$\binom{n}{a} = \frac{n!}{(n-a)!a!}$$

Identification of symmetric functions

- **Sufficient condition for symmetry:**

  1) **Column sums are all the same**
  2) **Each a-number "a", must occur a binomial coefficient number of times**

$$\frac{3!}{(3-1)!1!} = 3$$ **3 variables, 1-number**

$$\frac{3!}{(3-3)!3!} = 1$$ **3 variables, 3-number**

Identification of Symmetry Example

- **f=S(1, 2, 4, 7) - Canonical Form – Sum of Minterms - Symmetric**

x	y	z	
0	0	1	1
0	1	0	1
1	0	0	1
1	1	1	3
2	2	2	

$$\binom{3}{1} = \frac{3!}{(3-1)!1!} = 3$$

$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = 1$$

- **g= S(1, 2, 4, 5) - Canonical Form – Sum of Minterms – NOT Symmetric**

x	y	z	
0	0	1	1
0	1	0	1
1	0	0	1
1	0	1	2
2	1	2	

$$\binom{3}{1} = \frac{3!}{(3-1)!1!} = 3$$

$$\binom{3}{2} = \frac{3!}{3!(3-2)!} = 3$$

Identification of Symmetry Example 2

- **f(w,x,y,z)=S(0,1,3,5,8,10,11,12,13,15)**

w	x	y	z	a_i
0	0	0	0	0
0	0	0	1	1
0	0	1	1	2
0	1	0	1	2
1	0	0	0	1
1	0	1	0	2
1	0	1	1	3
1	1	0	0	2
1	1	0	1	3
1	1	1	1	4
6	4	4	6	

- **Column Sums are not Equal**
- **Not Totally Symmetric**

Identification of Symmetry Example 3

- **f(w,$\bar{x}$,$\bar{y}$,z)= S(3,5,6,7,9,10,11,12,13,14)**

w	$\bar{x}$	$\bar{y}$	z	a_i
0	0	1	1	2
0	1	0	1	2
0	1	1	0	2
0	1	1	1	3
1	0	0	1	2
1	0	1	0	2
1	0	1	1	3
1	1	0	0	2
1	1	0	1	3
1	1	1	0	3
6	6	6	6	

$$\binom{4}{2} = \frac{4!}{(4-2)!2!} = 6$$

$$\binom{4}{3} = \frac{4!}{(4-3)!3!} = 4$$

$$S_{2,3}(w, \bar{x}, \bar{y}, z)$$

<u>ALSO</u>

$$S_{1,2}(\bar{w}, x, y, \bar{z})$$

Column Sum Theorem

Theorem:

- **The equality of all column sums is NOT a sufficient condition for the detection of Total Symmetry**

Proof:

- **We prove this by contradiction. Consider the following function:**
- **f(w, x, y, z) = S(0,3,5,10,12,15)**
- **Clearly, it is NOT symmetric since a=2 is not satisfied, however, all column sums are the same.**

w	x	y	z	a_i
0	0	0	0	0
0	0	1	1	2
0	1	0	1	2
1	0	1	0	2
1	1	0	0	2
1	1	1	1	4
3	3	3	3	

$$\binom{4}{0} = 1 \qquad \binom{4}{2} = 6 \qquad \binom{4}{4} = 1$$

NOT Totally Symmetric!!!

Column Sum Check

- **Recall the Shannon Expansion Property**
  - **All co-factors of a symmetric function are also symmetric**
  - **When column sums are equal, expand about any variable**
  - **Consider f_w and $f_{w'}$**

w	x	y	z	a_i
0	0	0	0	0
0	0	1	1	2
0	1	0	1	2
1	0	1	0	2
1	1	0	0	2
1	1	1	1	4
3	3	3	3	

w	x	y	z	a_i
0	0	0	0	0
0	0	1	1	2
0	1	0	1	2
	1	1	2	

w	x	y	z	a_i
1	0	1	0	1
1	1	0	0	1
1	1	1	1	3
	2	2	1	

- **Cofactors NOT symmetric since column sums are unequal**
- **However, can complement variables to obtain symmetry**

{*x*¢, *y*¢} OR {*z*¢}

Total Symmetry Algorithm

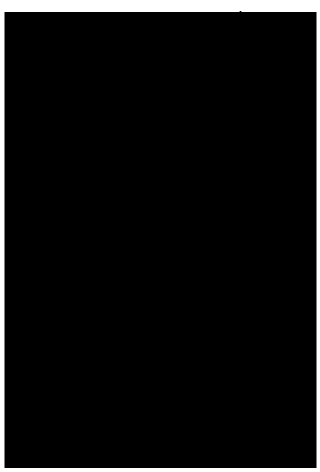
1) **Compute Column Sums**
   a) **if >2 column sum values ® *NOT SYMMETRIC***
   b) **if =2 compare the total with # rows**
      - **if same complement columns with smaller column sum**
      - **else NOT SYMMETRIC**
   c) **if =1, compare to ½ # of rows**
      - **if equal, go to step 2**
      - **if not equal, go to step 3**
2) **Compute Row Sums (*a*-numbers), check for correct values** $\binom{n}{a}$
   a) **if values are correct, then *SYMMETRY* detected**
   b) **if values are incorrect, then *NOT SYMMETRIC***

Total Symmetry Algorithm (cont)

3) **Compute Row Sums, check for correct numbers**
   - **if they are correct → *SYMMETRIC***
   - **else, expand f about any variable and go to step 1 for each cofactor**

Example

- **f(w,x,y,z)=S(0,1,3,5,8,10,11,12,13,15)**



- **Column sums are not equal**
- **There are two different sums (6 and 4)**
- **Compare the total (6+4=10) with the # of rows (10)**
- **They are the same ® complement columns with smaller column sum**

Example

- **f(w,x,y,z)=S(0,1,3,5,8,10,11,12,13,15)**

w	x'	y'	z	a_i
0	1	1	0	
0	1	1	1	
0	1	0	1	
0	0	1	1	
1	1	1	0	
1	1	0	0	
1	1	0	1	
1	0	1	0	
1	0	1	1	
1	0	0	1	
6	6	6	6	

- **a-numbers:**

$$\frac{4!}{(4-2)!2!} = 6 \qquad \frac{4!}{(4-3)!3!} = 4$$