Midterm Solutions Version B

Problem 1

a) Consider the following sets:

$K = \{x|(x^2 - x - 2)(x - 3)^2(x - 4) = 0\}$

$L = \{x|x$ is a letter in the word "know"$\}$

$M = \{x|x \in \mathbf{Z}\}$, and

$N = \{x|x \in [3, 7]\}$.

i) Compute the power set of L, $P(L)$.

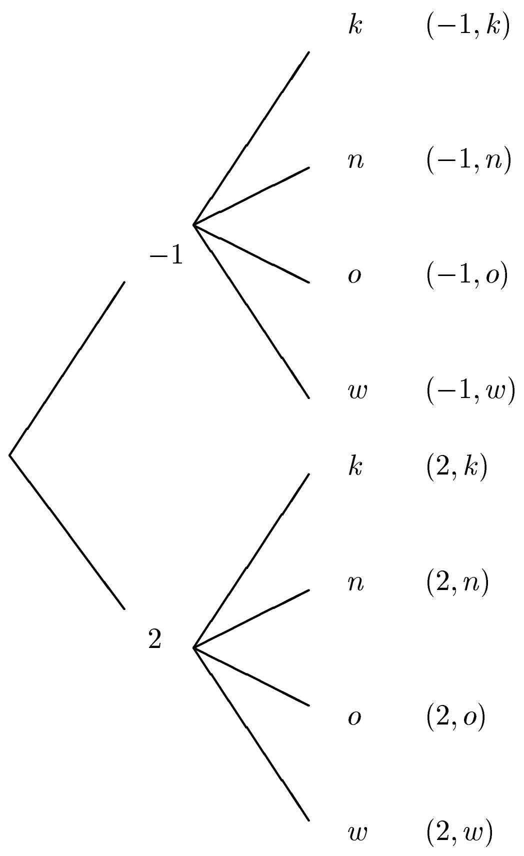
Solution $L = \{-1, 2, 3, 4\}$ So

$P(L) = \{L, \emptyset, \{k\}, \{n\}, \{o\}, \{w\}, \{k, n\}, \{k, o\}, \{k, w\}, \{n, o\}, \{n, w\}, \{o, w\},$
$\{k, n, o\}, \{k, n, w\}, \{k, o, w\}, \{n, o, w\}\}$

ii) Construct a tree diagram for $(K \cap (M \cap N^c)) \times L$.

Solution

Since $(M \cap N^c)$ are all integers exept $3, 4, 5, 6, 7$ so $(K \cap (M \cap N^c)) \times L = \{-1, 2\}$ we have



b) Prove that for all sets A and B, $((B^c \backslash A) \cup B^c)^c = B$.

Hint: You might want to use Theorem 1.2 in your book.

Solution

$$\begin{aligned}
((B^c \backslash A) \cup B^c)^c &= ((B^c \cap A^c) \cup B^c)^c \\
&= ((B \cup A)^c \cup B^c)^c \\
&= (B \cup A) \cap B \\
&= B
\end{aligned}$$

Problem 2 A pizza parlor has 10 different toppings for pizza.(They have an unlimited supply of pizzas and of each topping.) A party of 8 orders 8 pizzas, one topping per pizza. What is the probability that at least 2 of them got the same pizza?

Solution Let A be the event that at least 2 order the same topping. A^c is the event that **none** order the same one.

So $P(A) = 1 - P(A^c)$ Since there are 8 people and 10 different toppings, there are 10^8 ways in which the 8 people can get a topping. On the other hand, if the 8 people are to have different toppings the first one has 10 choices, the second one 9 ..., and the last one 3. So the number of ways they can pick a different topping is

$$P(10, 8) = 10 \cdot 9 ... \cdot 4 \cdot 3 = \frac{10!}{(10-8)!}$$

and the probability that at least 2 of them got the same topping is

$$P(A) = 1 - \frac{10!}{(10-8)!10^8}$$

Problem 3 Suppose you are given a green die and a blue die. Let,

$$\begin{aligned} A &= \text{The event that you roll a one} \\ B &= \text{The event that you roll a two} \\ &\vdots \\ F &= \text{The event that you roll a six} \end{aligned}$$

Now, suppose the green die is weighted such that $P(C) = 0$, $P(D) = 2/6$, and $P(A) = P(B) = P(E) = P(F) = 1/6$.

Also, the blue die is weighted such that $P(F) = 3/6$, $P(A) = P(B) = 0$, and $P(C) = P(D) = P(E) = 1/6$.

a) If you roll the green die twice what is the probability that you first roll a six then roll a four?

Solution

The probability of rolling a six with the green die is $1/6$. The probability of rolling a four with the green die is $2/6$.

Since the two event are independent we get:

$$p = \frac{1}{6} \cdot \frac{2}{6} = \frac{2}{36} = \frac{1}{18}$$

b) If you roll the dice at the same time, what is the probability that their sum is less than five, given that the green die does not come up as a two ?

Solution

The green die can come up $G = \{1, 2, 4, 4, 5, 6\}$ and

The blue die can come up $B = \{3, 4, 5, 6, 6, 6\}$

But we are given that the green cannot come up two. So $G* = \{1, 3, 5, 6\}$ so the sample space has 6×5 possible outcome. The event $A =$ "The sum is less than five" has only one possibility ($G*$ is 1 and B is 3)

So

$$P(A) = \frac{1}{30}$$

c) What is the probability of rolling the dice once and getting a sum of six given that the green die comes up as a four ?

Solution

The answer is 0 since the probability of rolling a 2 on the blue is 0.

Problem 4 Suppose you have six different coins in your wallet. Two of the coins are fair (that is $P(H) = P(T) = 1/2$). The other four coins are fake, such that $P(H) = 1$ and $P(T) = 0$.

the sample space consist of four events: We can have either a fair or fake coin, and then obtain either Head or tail. So

$$\begin{aligned} P(\text{Fair} \cap \text{Tail}) &= \frac{2}{6} \cdot \frac{1}{2} = \frac{1}{6} \\ P(\text{Fair} \cap \text{Head}) &= \frac{2}{6} \cdot \frac{1}{2} = \frac{1}{6} \\ P(\text{Fake} \cap \text{Tail}) &= \frac{4}{6} \cdot 0 = 0 \\ P(\text{Fake} \cap \text{Head}) &= \frac{4}{6} \cdot 1 = \frac{4}{6} \end{aligned}$$

a) What is the probability you take a coin from your wallet, toss it and it comes up heads?

Solution This is asking $P(\text{Head})$ which is:

$$\begin{aligned} P(\text{Head}) &= P((\text{Fake} \cap \text{Head}) \cup (\text{Fair} \cap \text{Head})) && (1) \\ &= P(\text{Fake} \cap \text{Head}) + P(\text{Fair} \cap \text{Head}) && (2) \\ &= \frac{4}{6} \cdot 1 + \frac{2}{6} \cdot \frac{1}{2} && (3) \\ &= \frac{5}{6} && (4) \end{aligned}$$

b) Suppose you take out one coin and toss it, getting a head. What is the probability the coin selected is fair ?

Solution Here we are asked $P(Fair|Head)$ which is

$$P(Fair|Head) = \frac{P(\text{Fair} \cap \text{Head})}{P(Head)} = \frac{\frac{1}{6}}{\frac{2}{3} + \frac{1}{6}} = \frac{1}{5}$$

Problem 5 Suppose you do not speak a foreign language, however for your survival you must answer 20 multiple choice questions in that language and score above 0. Each question has three possible choices a), b) or c). You must guess the answer for each question. Your score will be determined as follows:

A correct answer get +3 points
A wrong answer get -1 points.

a) Do you expect to survive ?

Solution Let X_i be your score at each question. X_i can only take on two value right($X = +3$) or wrong and ($X = -1$) and the probability of getting it right is $p = 1/3$ so the distribution of X_i is given by

x_i	3	-1
$f(x_i)$	$\frac{1}{3}$	$\frac{2}{3}$

the Expected value of X_i is Therefore $E(X_i) = 3 \cdot \frac{1}{3} - 1 \cdot \frac{2}{3} = \frac{1}{3}$

$$E(X) = \sum_{i=1}^{20} E(X_i) = 20 \cdot \frac{1}{3} = \frac{20}{3}$$

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Yes I do expect to survive.

b) Determine the variance and standard deviation.

Solution

$$E(X_i^2) = 3^2 \cdot \frac{1}{3} + (-1)^2 \cdot \frac{2}{3} = \frac{11}{3}$$

$$\begin{aligned} Var(X_i) &= E(X_i^2) - (E(X_i))^2 \\ &= \frac{11}{3} - \left(\frac{1}{3}\right)^2 \\ &= \frac{32}{9} \end{aligned}$$

Since you choose at random the X_i are independent and

$$\begin{aligned} Var(X) &= \sum_{i=1}^{20} Var(X_i) \\ &= 20 \cdot \frac{32}{9} \\ &= \frac{640}{9} \end{aligned}$$