Sample Quiz 7

1. Suppose the Tower of Hanoi has 5 poles instead of three. A disk can be transferred from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk.

   Let S_n be the minimum number of moves to transfer the entire tower of *n* disks from the leftmost to the rightmost pole.

   Find $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$

   Let the notation m[*n*,*s*,*d*] indicate the minimum number of moves to transfer *n* disks from pole *s* to pole *d*.

   S_1 = m[1,A,E] = 1
   S_2 = m[1,A,D] + S_1 + m[1,D,E] = 1+1+1 = 3
   S_3 = m[1,A,C] + S_2 + m[1,C,E] = 1+3+1 = 5
   S_4 = m[1,A,B] + S_3 + m[1,B,E] = 1+5+1 = 7
   S_5 = m[2,A,B] + S_3 + m[2,B,E] = 3+5+3 = 11
   S_6 = m[3,A,B] + S_3 + m[3,B,E] = 5+5+5 = 15
   S_7 = m[4,A,B] + S_3 + m[4,B,E] = 7+5+7 = 19
   S_8 = m[4,A,B] + m[2,A,C] +S_2 + m[2,C,E] + m[4,B,E] = 7+3+3+3+7 = 23

2. Prove the following for the Fibonacci Sequence $F_0, F_1, F_2, \ldots$

   $$F_{k+1}^2 + F_{k-1}^2 = 2\ F_{k+1}\ F_{k-1} + F_k^2 \ , \quad k \geq 1$$

   We know that $F_{k+1} = F_k + F_{k-1}$.

   Solving for F_k we get $F_k = F_{k+1} - F_{k-1}$.

   Squaring both sides, we get $F_k^2 = F_{k+1}^2 - 2\ F_{k+1}\ F_{k-1} + F_{k-1}^2$

   Now the proof is just a substitution away:

   $$F_{k+1}^2 + F_{k-1}^2 = 2\ F_{k+1}\ F_{k-1} + F_k^2 \ , \quad k \geq 1$$

   $F_{k+1}^2 + F_{k-1}^2 = 2\ F_{k+1}\ F_{k-1} + F_{k+1}^2 - 2\ F_{k+1}\ F_{k-1} + F_{k-1}^2$ by substitution

   $F_{k+1}^2 + F_{k-1}^2 = F_{k+1}^2 + F_{k-1}^2$

3. Use iteration to guess an explicit formula for the following recursively defined sequence. Use the formulas from Epp (4.2) to simplify your answers whenever possible.

$b_k = 3\ b_{k-1} + 4$, for $k \geq 1$
$b_0 = 2$

b_k
(by defn) = $3\ b_{k-1} + 4$

(by subst) = $3\ [\ 3\ b_{k-2} + 4] + 4$

(by simpl) = $(3^2)b_{k-2} + (3)4 + 4$

(by subst) = $(3^2)[\ 3\ b_{k-3} + 4] + (3)4 + 4$

(by simpl) = $(3^3)b_{k-4} + (3^2)4 + (3)4 + 4$

(by subst) = $(3^3)[\ 3\ b_{k-4} + 4] + (3^2)4 + (3)4 + 4$

(by simpl) = $(3^4)b_{k-4} + (3^3)4 + (3^2)4 + (3)4 + 4$

$\vdots$

$b_k = (3^k)b_{k-k} + (3^{k-1})4 + (3^{k-2})4 + ...+ (3^2)4 + (3)4 + 4$

$b_k = (3^k)b_0 + 4\sum_{i=0}^{k-1} 3^i$

$b_k = (3^k)(2) + 4\frac{3^{(k-1)+1}-1}{3-1}$

$b_k = 2\ (3^k) + 2\ (\ 3^k\ \text{-1}\ )$

$b_k = 2\ (3^k) + 2\ (3^k)\ \text{-2}$

$b_k = 4\ (3^k)\ \text{-2}$