

Sample Quiz 7

1. Suppose the Tower of Hanoi has 5 poles instead of three. A disk can be transferred from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk.

Let S_n be the minimum number of moves to transfer the entire tower of n disks from the leftmost to the rightmost pole.

Find $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$

Let the notation $m[n,s,d]$ indicate the minimum number of moves to transfer n disks from pole s to pole d .

$$S_1 = m[1,A,E] = 1$$

$$S_2 = m[1,A,D] + S_1 + m[1,D,E] = 1+1+1 = 3$$

$$S_3 = m[1,A,C] + S_2 + m[1,C,E] = 1+3+1 = 5$$

$$S_4 = m[1,A,B] + S_3 + m[1,B,E] = 1+5+1 = 7$$

$$S_5 = m[2,A,B] + S_3 + m[2,B,E] = 3+5+3 = 11$$

$$S_6 = m[3,A,B] + S_3 + m[3,B,E] = 5+5+5 = 15$$

$$S_7 = m[4,A,B] + S_3 + m[4,B,E] = 7+5+7 = 19$$

$$S_8 = m[4,A,B] + m[2,A,C] + S_2 + m[2,C,E] + m[4,B,E] = 7+3+3+3+7 = 23$$

2. Prove the following for the Fibonacci Sequence $F_0, F_1, F_2, \dots$

$$F_{k+1}^2 + F_{k-1}^2 = 2 F_{k+1} F_{k-1} + F_k^2, \quad k \geq 1$$

We know that $F_{k+1} = F_k + F_{k-1}$.

Solving for F_k we get $F_k = F_{k+1} - F_{k-1}$.

Squaring both sides, we get $F_k^2 = F_{k+1}^2 - 2 F_{k+1} F_{k-1} + F_{k-1}^2$

Now the proof is just a substitution away:

$$F_{k+1}^2 + F_{k-1}^2 = 2 F_{k+1} F_{k-1} + F_k^2, \quad k \geq 1$$

$$F_{k+1}^2 + F_{k-1}^2 = 2 F_{k+1} F_{k-1} + F_{k+1}^2 - 2 F_{k+1} F_{k-1} + F_{k-1}^2 \text{ by substitution}$$

$$F_{k+1}^2 + F_{k-1}^2 = F_{k+1}^2 + F_{k-1}^2$$

3. Use iteration to guess an explicit formula for the following recursively defined sequence. Use the formulas from Epp (4.2) to simplify your answers whenever possible.

$$b_k = 3 b_{k-1} + 4, \text{ for } k \geq 1$$

$$b_0 = 2$$

$$b_k$$

$$(\text{by defn}) = 3 b_{k-1} + 4$$

$$(\text{by subst}) = 3 [3 b_{k-2} + 4] + 4$$

$$(\text{by simpl}) = (3^2) b_{k-2} + (3)4 + 4$$

$$(\text{by subst}) = (3^2) [3 b_{k-3} + 4] + (3)4 + 4$$

$$(\text{by simpl}) = (3^3) b_{k-4} + (3^2)4 + (3)4 + 4$$

$$(\text{by subst}) = (3^3) [3 b_{k-4} + 4] + (3^2)4 + (3)4 + 4$$

$$(\text{by simpl}) = (3^4) b_{k-4} + (3^3)4 + (3^2)4 + (3)4 + 4$$

$\vdots$

$$b_k = (3^k) b_{k-k} + (3^{k-1})4 + (3^{k-2})4 + \dots + (3^2)4 + (3)4 + 4$$

$$b_k = (3^k) b_0 + 4 \sum_{i=0}^{k-1} 3^i$$

$$b_k = (3^k)(2) + 4 \frac{3^{(k-1)+1} - 1}{3 - 1}$$

$$b_k = 2 (3^k) + 2 (3^k - 1)$$

$$b_k = 2 (3^k) + 2 (3^k) - 2$$

$$b_k = 4 (3^k) - 2$$