Quiz 7 Solutions

Problem 1 Consider the following sequence:

$$a_k = a_{k-1} + 3^k \qquad \text{for all } k \geq 2 \qquad a_1 = 1$$

i) Use iteration to derive an explicit formula for the sequence
Solution

$$\begin{aligned}
a_1 &= 1 = 3^0 \\
a_2 &= 3^0 + 3^2 \\
a_3 &= 3^0 + 3^2 + 3^3 \\
&\vdots \\
a_n &= 3^0 + 3^2 + ... + 3^n \\
3^1 + a_n &= 3^0 + 3^1 + 3^2 + ... + 3^n \\
3^1 + a_n &= \sum_{i=0}^{n} 3^i \\
3^1 + a_n &= \frac{3^{n+1} - 1}{3 - 1} \\
a_n &= \frac{3^{n+1} - 7}{2}
\end{aligned}$$

ii) Use induction to prove that your explicit formula holds for all n.
Solution

base step $k = 2$

$$a_2 = \frac{3^{2+1} - 7}{2} = 10$$

induction step

Assume $a_n = \frac{3^{n+1}-7}{2}$
To show $a_{n+1} = \frac{3^{n+2}-7}{2}$
Proof

$$\begin{array}{lll} a_{n+1} & = a_n + 3^{n+1} & \text{by part a)} \\ a_{n+1} & = \frac{3^{n+1}-7}{2} + \frac{2}{2}3^{n+1} & \text{by induction hypothesis} \\ a_{n+1} & = \frac{(1+2)3^{n+1}-7}{2} & \text{by algebra} \\ a_{n+1} & = \frac{3^{n+2}-7}{2} & \end{array}$$

Problem 2 The recursive relation $F_k = F_{k-1} + F_{k-2}$ satisfies the Fibonacci sequence.

i) Explain why $F_{k+2} = (F_{k-1} + F_{k-2}) + (F_k + F_{k_1})$
Solution

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+2} &= F_k + F_{k-1} + F_k \\ F_{k+2} &= F_{k-1} + F_{k-2} + F_{k-1} + F_k \end{aligned}$$

ii) Prove that $F_k^2 = 2F_kF_{k-2} + F_k(F_{k-4} + F_{k-5}$.
Solution

$$\begin{aligned} 2F_kF_{k-2} + F_k(F_{k-4} + F_{k-5}) &= 2F_kF_{k-2} + F_kF_{k-3} \\ &= F_k(2F_{k-2} + F_{k-3} \\ &= F_k(F_{k-2} + (F_{k-2} + F_{k-3})) \\ &= F_k(F_{k-2} + F_{k-1}) \\ &= F_k^2 \end{aligned}$$

Problem 3 This problem is almost exactly the one fully explain p 431. The only change is the number of offspring

1) Find the recurrence relation for r_k in terms of r_{k-1} and r_{k-2} for all $k \geq 1$.

Solution

$$r_k = r_{k-1} + 4r_{k-2} \qquad \text{for all integers} \quad k \geq 2$$

for detailed explanation see Epp Example 8.1.6

2) Find an explicit formula for r_n for $n \geq 2$ and verify it via induction.

Solution Let $r_k = r_{k-1} + 4r_{k-2}$

with $r_0 = 1$ and $r_1 = 1$ the caracteristic equation is:

$$t^2 - t - 4 = 0$$

$$t_1 = \frac{1 + \sqrt{1 - 4(-4)}}{2} = \frac{1 + \sqrt{17}}{2}$$

and

$$t_2 = \frac{1 - \sqrt{1 - 4(-4)}}{2} = \frac{1 - \sqrt{17}}{2}$$

thus we get

$$r_n = C \cdot (t_1)^n + D(t_2)^n$$

and

$$\begin{aligned} r_0 &= C \cdot (t_1)^0 + D \cdot (t_2)^0 = C + D = 1 \\ r_1 &= C \cdot (t_1)^1 + D \cdot (t_2)^1 = Ct_1 - Dt_2 = 1 \end{aligned}$$

which gives(after tedious algebra) $D = \frac{t_1 - 1}{t1 - t2} = \frac{1}{2}\left(1 - \frac{\sqrt{17}}{17}\right)$ and $C = 1 - D = \frac{1}{2}\left(1 + \frac{\sqrt{17}}{17}\right)$. Hence, (just for the fun ...)

$$r_n = \frac{1}{2}\left(1 + \frac{\sqrt{17}}{17}\right) \cdot \left(\frac{1 + \sqrt{17}}{2}\right)^n + \frac{1}{2}\left(1 - \frac{\sqrt{17}}{17}\right) \cdot \left(\frac{1 - \sqrt{17}}{2}\right)^n$$

for all $n \geq 2$.

proof by induction

base step $n = 2$ recall that $r_2 = 1 + 4(1) = 5$

I worked out the following algebra

$$r_2 = \frac{1}{2}\left(1 + \frac{\sqrt{17}}{17}\right) \cdot \left(\frac{1 + \sqrt{17}}{2}\right)^2 + \frac{1}{2}\left(1 - \frac{\sqrt{17}}{17}\right) \cdot \left(\frac{1 - \sqrt{17}}{2}\right)^2$$

and got indeed $r_2 = 5$ (details upon request during philippe O. H.)

induction step

Assume:

$$r_n = C \cdot (t_1)^n + D(t_2)^n$$

To show:

$$r_{n+1} = C \cdot (t_1)^{n+1} + D(t_2)^{n+1}$$

The algebra for the proof is very messy, I 'm still working on it.