Quiz 7 Solutions

Problem 1 Consider the following sequence:

$$a_k = a_{k-1} + 2^k \qquad \text{for all } k \geq 2 \qquad a_1 = 1$$

i) Use iteration to derive an explicit formula for the sequence
Solution

$$\begin{aligned}
a_1 &= 1 = 2^0 \\
a_2 &= 2^0 + 2^2 \\
a_3 &= 2^0 + 2^2 + 2^3 \\
&\vdots \\
a_n &= 2^0 + 2^2 + ... + 2^n \\
2^1 + a_n &= 2^0 + 2^1 + 2^2 + ... + 2^n \\
2^1 + a_n &= \sum_{i=0}^{n} 2^i \\
2^1 + a_n &= \frac{2^{n+1} - 1}{2 - 1} \\
2 + a_n &= 2^{n+1} - 1 \\
a_n &= 2^{n+1} - 3
\end{aligned}$$

ii) Use induction to prove that your explicit formula holds for all n.
Solution

base step $k = 2$

$$a_2 = 2^{2+1} - 3 = 5$$

induction step

Assume $a_n = 2^{n+1} - 3$
To show $a_{n+1} = 2^{n+2} - 3$
Proof

$$
\begin{aligned}
a_{n+1} &= a_n + 2^{n+1} && \text{by part a)} \\
a_{n+1} &= 2^{n+1} - 3 + 2^{n+1} && \text{by induction hypothesis} \\
a_{n+1} &= 2\left(2^{n+1}\right) - 3 && \text{by algebra} \\
a_{n+1} &= 2^{n+2} - 3
\end{aligned}
$$

Problem 2 The recursive relation $F_k = F_{k-1} + F_{k-2}$ satisfies the Fibonacci sequence.

i) Explain why $F_{k+2} = F_k + (F_k + F_{k-1})$
Solution

$$
\begin{aligned}
F_{k+2} &= F_{k+1} + F_k && \text{by letting } k+2 \text{ be } k \text{ in the Finobacci formula} \\
F_{k+2} &= F_k + F_{k-1} + F_k && \text{by letting } k+1 \text{ be } k \text{ in the Finobacci formula}
\end{aligned}
$$

ii) Prove that $F_k^3 = 2F_k(F_{k-1} + F_{k-2})F_{k-2} + F_k^2 F_{k-3}$.
Solution

$$
\begin{aligned}
2F_k(F_{k-1} + F_{k-2})F_{k-2} + F_k^2 F_{k-3} &= 2F_k F_k F_{k-2} + F_k^2 F_{k-3} \\
&= F_k^2(2F_{k-2} + F_{k-3}) \\
&= F_k^2(F_{k-2} + (F_{k-2} + F_{k-3})) \\
&= F_k^2(F_{k-2} + F_{k-1}) \\
&= F_k^3
\end{aligned}
$$

Problem 3 A pizza is cut into n distinct slices. Each slice has either mushrooms, bellpeppers, tomatoes or garlic such that no two adjacent slices have the same toppings.

Let S_n be the number of ways to make the pizza.

1) Find the recurrence relation for S_k in terms of S_{k-1} and S_{k-2} for all $k \geq 4$.

Solution Let k be an integer with $k \geq 3$, call the k slice of the pizza $1, 2, 3, ..., k$, and suppose that the values of $S_1, S_2, ..., S_{k-1}$ are known.

Note that $S_1 = 4$, $S_2 = 4 \cdot 3 = 12$.

Case1, slices (or sectors) 1 and 3 have the same topping: In this case $k > 3$ because otherwise slices 1 and 3 would be both adjacent and with the same topping which is not allowed. Imagine slices 1 through 3 are made into a single unit with same topping of sectors 1 and 3. Then the pizza would contain a total of $k - 2$ sectors:1-3,4,5,...,k, and he number of ways to make the pizza would be S_{k-2}. Each of these ways corresponds to exactly 3 ways to make the pizza when sectors 1-3 are not united, for that is the number of choices of topping to choose for sector 2 to contrast with sectors 1 and 3. Hence there are $3S_{k-2}$ ways to make the pizza in this case.

Case 2, Sectors 1 and 3 are of different topping: Imagine shrinking sector 2 to nothing. Then there would be S_{k-1} way to make the pizza. Now imagine expanding sector 2 back to its original size and giving it its original topping. Since there are2 ways that sector 2 could be

topped that would contrast with topping of sectors 1 and 3, each ways to top the pizza leaving sector 2 out correspond to exactely 2 ways to make the pizza when sector 2 is there. Hence, there are $2S_{k-1}$ ways to make the pizza in this case.

If $k = 3$, then case 1 cannot occurs and $S_3 = 2S_2 = 24$. If $k \geq 4$, then the total number of ways to make the pizza is the sum of case 1 and 2.

$$S_k = 2S_{k-1} + 3S_{k-2} \qquad \text{for all integers} \quad k \geq 4$$

2) Find an explicit formula for S_n for $n \geq 2$ and verify it via induction.

Solution Let $T_n = S_{n+2}$ for all $n \geq 0$ so that for all $k \geq 2$,

$$T_k = 2T_{k-1} + 3T_{k-2}$$

with $T_0 = 12$ and $T_1 = 24$ the caracteristic equation is:

$$t^2 - 2t - 3 = 0$$

$$(t+1)(t-3) = 0$$

thus we get

$$T_n = C \cdot 3^n + D(-1)^n$$

and

$$\begin{aligned} T_0 &= C \cdot 3^0 + D \cdot (-1)^0 = C + D = 12 \\ T_1 &= C \cdot 3^1 + D \cdot (-1)^1 = 3C - D = 24 \end{aligned}$$

which gives $C = 9$ and $D = 3$. Hence, $T_n = 9 \cdot 3^n + 3 \cdot (-1)^n$ for all $n \geq 0$ thus

$$S_n = T_{n-2} = 9 \cdot 3^{n-2} + 3 \cdot (-1)^{n-2}$$

for all $n \geq 2$.

proof by induction

base step $n = 2$ is trivial

induction step

Assume:

$$S_n = 9 \cdot 3^{n-2} + 3 \cdot (-1)^{n-2}$$

To show:

$$S_{n+1} = 9 \cdot 3^{n-1} + 3 \cdot (-1)^{n-1}$$

$$\begin{aligned} S_{k+1} &= 2S_k + 3S_{k-1} && \text{by part a)} \\ &= 2(9)3^{n-2} + 2(3)(-1)^{n-2} + 3(9)3^{n-3} + 3(3)(-1)^{n-3} && \text{by induction hypothesis} \\ &= 2(9)3^{n-2} + 2(3)(-1)^{n-2} + (9)3^{n-2} + 3(-1)(3)(-1)^{n-2} && \text{by basic algebra} \\ &= (2+1)(9)3^{n-2} + (2-3)(3)(-1)^{n-2} && \text{by basic algebra} \\ &= (3)(9)3^{n-2} + (-1)(3)(-1)^{n-2} \\ &= (9)3^{n-1} + (3)(-1)^{n-1} && \text{as was to be shown.} \end{aligned}$$