Quiz 5 Solutions

Problem 1 Two teams are playing in the 2nd round of the platoffs. Assume the rules have changed such that the first team to win 2 games in a row or 4 games total adavances to the next round. If the probability that team A wins any single game is 1/3, find the expected number of games the teams will play in this round.

Solution Let X be a random variable that assign the number of games. Let L be "team A looses and W team A wins.

$$\begin{aligned}
P(X=2) &= P(WW)+P(LL)=\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2=\frac{5}{3^2}\\
P(X=3) &= P(WLL)+P(LWW)=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2=\frac{6}{3^3}\\
P(X=4) &= P(LWLL)+P(WLWW)=\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)=\frac{10}{3^4}\\
P(X=5) &= P(WLWLL)+P(LWLWW)=\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3=\frac{12}{3^5}\\
P(X=6) &= P(WLWLWW)+P(LWLWLL)=\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2=\frac{20}{3^6}\\
P(X=7) &= P(WLWLWLL)+P(LWLWLWW)+P(WLWLWLW)+P(LWLWLWL)\\
&= 2\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^3+2\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3=\frac{48}{3^7}
\end{aligned}$$

So

$$E(X)=2\left(\frac{5}{3^2}\right)+3\left(\frac{6}{3^3}\right)+4\left(\frac{10}{3^4}\right)+5\left(\frac{12}{3^5}\right)+6\left(\frac{20}{3^6}\right)+7\left(\frac{48}{3^7}\right)=\frac{2068}{3^6}$$

Therefore they expect to play around 3 games.

Problem 2 Consider the following joint distribution table:

$X\backslash Y$	-3	-1	2	3	Sum
1	.1	.3	0	.1	.5
2	0	.1	.2	.2	.5
Sum	.1	.4	.2	.3	

1) Find $E(X)$ and $E(Y)$.

Solution

$$E(X) = 1 \cdot .5 + 2 \cdot .5 = \frac{3}{2}$$

$$E(Y) = -3 \cdot .1 - 1 \cdot .4 + 2 \cdot .2 + 3 \cdot .3 = \frac{3}{5}$$

2) Find the $Cov(X, Y)$.

Solution

$$E(XY) = (-3)(1)(.1) + (-1)(1)(.3) + (3)(1)(.1) + (2)(-1)(.1) + (2)(2)(.2) + (3)(2)(.2) = 1.5$$

So

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 1.5 - (1.5)(.6) = .6$$

3) Find the standard deviation of X and Y and $\rho(X, Y)$.

Solution

$$E(X^2) = (1)^2(.5) + (2)^2(.5) = \frac{5}{2}$$

$$Var(X) = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{1}{4}$$

$$\sigma_x = \sqrt{Var(X)} = \frac{1}{2}$$

$$E(Y^2) = (-3)^2(.1) + (-1)^2(.4) + (2)^2(.2) + (3)^2(.3) = \frac{25}{4}$$

$$Var(Y) = \frac{25}{4} - \left(\frac{3}{5}\right)^2 = \frac{111}{25}$$

$$\sigma_y = \sqrt{Var(Y)} = \frac{\sqrt{111}}{25}$$

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_x \sigma_y} = \frac{1.5}{\frac{\sqrt{111}}{50}} = \frac{75}{\sqrt{111}}$$

4) Are X and Y independent?

Solution

No, because $Cov(X, Y) \neq 0$. Please note that if you find a $Cov(X, Y) = 0$ for another problem it does **NOT** imply that the events are independent.

Problem 3 Let X be a continuous random variable with distribution

$$f(x) = \begin{cases} 1/8x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

1) Find the probability that $x \leq 2.5$.

Solution

$$\int_0^{2.5} \frac{x}{8} dx = \frac{1}{8} \frac{(2.5)^2}{2} = \frac{25}{64}$$

2) Plot the graph of the cumulative distribution function.

Solution

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{16} & \text{for } 0 \leq x \leq 4 \\ 1 & x > 4 \end{cases}$$