CSE 20, Discrete Mathematics, Winter 1995, Final Exam

Answer 6 problems as directed. Specify on the front of your blue book which problems you want graded. Each problem is worth 5 points.

Logic1 You know the following facts:

1. If x divides y evenly and y divides z evenly, then x divides z evenly.
2. If x does not divide y evenly, then $r \neq 0$.
3. If $r \neq 0$, then $r \in S$.
4. If $r \in S$, then x is not the smallest element of S.
5. x is the smallest element of S.
6. y divides z evenly.

Use propositional logic to express these facts using the underlying variables

1. $XY \iff x$ divides y evenly,
2. $YZ \iff y$ divides z evenly,
3. $XZ \iff x$ divides z evenly,
4. $R \iff r \neq 0$,
5. $RS \iff r \in S$,
6. and $SX \iff x$ is the smallest element of S.

Then prove from them that x divides z evenly.

Logic2 Let $f, g : Z^+ \rightarrow Z^+$ be two functions. We write $f \in \Theta(g)$ if and only if there are real numbers $c, d > 0$ so that $dg(x) \leq f(x) \leq cg(x)$ for all positive integers x. Prove that if $f \in \Theta(g)$, then $g \in \Theta(f)$.

Ind1 Consider the following program that makes a recursive call to itself:

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Function: Exp(X,Y) ; Input: X, Y Integer;
IF Y=0 THEN Return 1;
        ELSE
        IF (Y MOD 2) =0 THEN Z:=Exp(X , Y DIV 2);Return Z*Z;
                    ELSE Return Exp(X,Y-1)*X;
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Prove that for any integer value of $X, Y \geq 0$, $Exp(X,Y) = X^Y$. (Hint: do a proof by strong induction on Y.)

Ind2 Let S be a set of integers *closed under multiplication*, i.e. if $x \in S$ and $y \in S$, then $xy \in S$. (Some examples are the set of even integers, the set of odd integers, or the set of perfect squares.) Prove that if $x \in S$, then $x^k \in S$ for every $k \geq 1$.

Relations Let $S = \{0,1\}^5$, the set of all 5 element string of 0's and 1's. Consider the relation : $wRw' \iff$ the number of 0's in w = the number of 0's in w'. Is R a partial order, equivalence relation, or neither? If it is a partial order, give a maximal anti-chain. If it is an equivalence relation, how many distinct equivalence classes are there? If neither, give counter-examples that show that R violates the conditions for partial-orders and equivalences.

Pigeon-Hole Principle Say you knew the following "facts": more than 90% of Americans have one of the 200 most common last names, and more than 90% of Americans have one of the 50 most common first names. Also, at least 80% of Americans are under 40 years old. There are at least 250,000,000 Americans. Prove from these "facts" that there are two Americans with the same (first and last) name who were born on the same day. (Hint: what's the largest number of Americans that there could be with uncommon first names , uncommon last names *OR* that are over 40?)