CSE 20, Discrete Mathematics, Fall, 1995
Midterm 1

All problems are weighted equally at 10 points each.

- Convert the following to a symbolic form, and give a proof that the assumptions imply the conclusion. (Comments in parentheses are just to make the story better, and have no logical import.)

  (Peter Gallindo, a member of the L. A. Bicycle Advisory Committee, makes the following claims in a letter to the L.A. Times :)

  1. If the MTA put bike racks on their busses, and employers provided a place to change, then the average commuter would bike to work one day a week.
  2. If commuters take the initiative, (they can convince their) employers to provide a place to change.
  3. The MTA won't install bike racks on their busses only if commuters do not take the initiative.
  4. If the average commuter would bike to work one day a week, then there would be significantly less pollutants in the air.

  He oncludes that, if commuters take the initiative, there will be significantly less pollutants in the air. Prove that this conclusion follows from his premises.

  Answer: Let $M$ stand for "The MTA puts bike racks on their busses", $E$ for "employers provide a place to change", $A$ for "the average commuter bikes to work one day a week", $C$ for "commuters take initiative". and $S$ for "there is a significant reduction in air pollutants".

  The statements should be formalized as:

  1. $(M \wedge E) \rightarrow A$
  2. $C \rightarrow E$
  3. $M \rightarrow I$. (Note: the sentence says 'only if" not "if and only if", so it is not an equivalence.)
  4. $A \rightarrow S$.

  We want to show : $C \rightarrow S$.

  1. $(M \wedge E) \rightarrow A$ (premise)
  2. $C \rightarrow E$ (premise)
  3. $M \rightarrow I$. (premise)
  4. $A \rightarrow S$. (premise)

5. Assume $C$
6. $E$ (5,2, MP)
7. $M$ (5,3, MT)
8. $M \wedge E$ (6,7, CA)
9. $A$ (8,1, MP)
10. $S$ (9,4, MP)
11. Therefore $C \rightarrow S$ (5,10, syllogism)

- Let $A$,$B$, and $C$ be sets of elements from a common universe $U$, You are given that $A \cap C \neq \emptyset$ and that $A \subseteq B$. Prove that $C \cap B \neq \emptyset$.

  Discussion: The empty set $\emptyset$ is the set with no elements. To say that a set is not the emptyset is to say that it has at least one element. Thus, our goal is to show $\exists x[x \in C \cap B]$. We are told that $\exists[x \in A \cap C]$ and that $\forall x[x \in A \rightarrow x \in B]$. Thus our strategy is: to find an $x$ element of $C \cap B$, use the element that we know is in $A \cap C$ That element is in $A$ and $C$, and by the subset relation, it is also in $B$. So it is in $C \cap B$. This is formalized as follows:

  1. $A \cap C \neq \emptyset$ (given)
  2. $A \subseteq B$ (given)
  3. $\exists x[x \in A \cap C]$ (1, defn of empty set)
  4. Choose $x_0 \in A \cap C$ (3, choice)
  5. $x_0 \in A \wedge x_0 \in C$ (4, defn of intersection)
  6. $\forall x[x \in A \rightarrow x \in B]$ (2, defn of subset)
  7. $x_0 \in A \rightarrow x_0 \in B$ (6, US)
  8. $x_0 \in B$ (5,7 MP)
  9. $x_0 \in B \wedge x_0 \in C$ (5,8, CA)
  10. $x_0 \in B \cap C$ (9, definition of intersection)
  11. $\exists x[x \in B \cap C$ (10, example)
  12. $B \cap C \neq \emptyset$ (11, definition of empty set)

- Say you are told the following facts:
  - A person is a niece or nephew of a second person if and only if that person is a child of one of the second person's siblings.
  - Every person is older than all of his or her children.
  - Alice is older than all of her siblings.

Formalize these facts using quantifiers, the predicates $N(x,y) \iff x$ is a niece or nephew of $y$, $S(x,y) \iff x$ is a sibling of $y$, $C(x,y) \iff x$ is a child of $y$, and the numerical function $A(x)$ to represent the age of person $x$.

Then use them to prove that Alice is older than any of her nieces or nephews.

Discussion: This is very obvious: Alice is older than her siblings, her siblings are older than their children, and so Alice is older than her siblings' children. The point is to get familiar with quantifiers and reasoning with quantifiers, in order to bring out the underlying logic in that argument. The biggest mistake people made is getting confused between the three types of objects: people, facts, and numbers (ages). The variables $x, y, ..$ represent people, the predicates $S(x,y)$ etc. are facts, and the function $A(x)$ is a number . So it makes no sense to say $S(x,y) > A(x)$, comparing a fact to a number, or to talk about $A(S(x,y))$.

The first sentence defines what it means to be someone's niece or nephew:

$N(x,y) \iff x$ is $y$'s niece or nephew $\iff x$ is the child of ONE of $y$'s siblings $\iff$ there is a sibling $z$ of $y$ so that $x$ is the child of $z \iff \exists z[S[y,z] \wedge C(x,z)]$. So the final traslation is: $N(x,y) \iff \exists z(S(y,z) \wedge C(x,z))]$.

The subject of the second statement is "every person", so it should be a universal stament $\forall x$ [something]. What is that something? "$x$ is older than every child of $x$" This is a general rule about children of $x$, or a general rule about people that only pertains if they are children of $x$. So the logical structure is $\forall x \forall y[C(y,x) \rightarrow y$ is older than $x]$, or since $y$ is older than $x$ means $A(y) > A(x)$, $\forall x \forall y[C(x,y) \rightarrow A(y) > A(x)]$. The third statement is similar, except instead of the subject being any person, the subject is Alice, and the general rule applies if $y$ is Alice's sibling: $\forall y[S(Alice,y) \rightarrow A(Alice) > A(y)]$.

The goal is to show a similar fact for neices or nephews of Alice: $\forall y[N(y, Alice) \rightarrow A(Alice) > A(y)]$.

We are trying to prove a universal statement, so we should use universal generalization, i.e. prove the inside for an arbitrary person $y_0$, i,e, Prove $N(y_0, Alice) \rightarrow A(Alice) > A(y_0)$. This sub-goal is an implication, so we should assume the hypothesis $N(Alice, y_0)$ and use that to prove $A(Alice) > A(y_0)$. To use the assumption, we'll first use the definition of $N(x,y)$ to find $z_0$, a sibling of Alice's that is a parent of $y_0$. We then use the two facts we are given to show that $A(Alice) > A(z_0) > A(y_0)$.

1. $\forall x \forall y[C(x,y) \rightarrow A(y) > A(x)]$. (given)
2. $\forall y[S(Alice,y) \rightarrow A(Alice) > A(y)]$. (given)

3. Let $y_0$ be an arbitrary person.
4. Assume $N(y_0, Alice)$
5. $\exists z[S(Alice, z) \wedge C(y_0, z)]$ (4, defn of niece or nephew)
6. Pick $z_0$ so that $S(Alice, z_0) \wedge C(y_0, z_0)$. (5, choice)
7. $S(Alice, z_0) \rightarrow A(Alice) > A(z_0)$ (2,US,$y = z_0$)
8. $A(Alice) > A(z_0)$ (6,7, MP)
9. $C(y_0, z_0) \rightarrow A(z_0) > A(y_0)$ (1,US,$x = y_0, y = z_0$)
10. $A(z_0) > A(y_0)$ (9,6, MP)
11. $A(Alice) > A(y_0)$ (10, 8, Algebra)
12. $N(y_0, Alice) \rightarrow A(Alice) > A(y_0)$ (4,11,syll.)
13. $\forall y[N(y, Alice) \rightarrow A(Alice) > A(y)]$ (3,12.UG)

Less formally, but still going through the above steps, we would say:

Let $y$ be any niece or nephew of Alice. By the definition of niece or nephew, some person $z$ must be both a sibling of Alice's and have $y$ as a child. Since Alice is older than any of her siblings, and $z$ is a sibling of Alice's. Alice's age is greater than $z$'s. Since $y$ is a child of $z$, and a child is younger than its parent, $z$'s age is greater than $y$'s. Thus, Alice's age is greater than $y$'s. Thus, Alice is older than any of her nieces or nephews.