CSE 20, Discrete Mathematics, Winter 1995, Final Exam

Answer 6 problems as directed. Specify on the front of your blue book which problems you want graded. Each problem is worth 5 points.

Logic1 You know the following facts:

1. If x divides y evenly and y divides z evenly, then x divides z evenly.
2. If x does not divide y evenly, then $r \neq 0$.
3. If $r \neq 0$, then $r \in S$.
4. If $r \in S$, then x is not the smallest element of S.
5. x is the smallest element of S.
6. y divides z evenly.

Use propositional logic to express these facts using the underlying variables

1. $XY \iff x$ divides y evenly,
2. $YZ \iff y$ divides z evenly,
3. $XZ \iff x$ divides z evenly,
4. $R \iff r \neq 0$,
5. $RS \iff r \in S$,
6. and $SX \iff x$ is the smallest element of S.

Then prove from them that x divides z evenly. Solution: Formalizing the givens we get:

- $(XY \wedge YZ) \rightarrow XZ$
- $\neg XY \rightarrow R$
- $R \rightarrow RS$
- $RS \rightarrow \neg SX$
- SX
- YZ

Our goal is to prove XZ

1. $(XY \wedge YZ) \rightarrow XZ$
2. $\neg XY \rightarrow R$
3. $R \rightarrow RS$
4. $RS \rightarrow \neg SX$
5. SX
6. YZ
7. $\neg RS$ (4.5 MT)
8. $\neg R$ (7,3 MT)
9. XY (8,2, MT)
10. $XY \wedge YZ$ (9.6. CA)
11. XZ (1,10, MP)

Logic2 Let $f, g : Z^+ \to Z^+$ be two functions. We write $f \in \Theta(g)$ if and only if there are real numbers $c, d > 0$ so that $dg(x) \leq f(x) \leq cg(x)$ for all positive integers x. Prove that if $f \in \Theta(g)$, then $g \in \Theta(f)$.

By the definition of $f \in \Theta(g)$, we know that there are $c_1, d_1 > 0$ so that $d_1 g(x) \leq f(x) \leq c_1 g(x)$ for all positive integers x. Then $g(x) \geq 1/c_1 f(x)$ and $g(x) \leq 1/d_1 f(x)$. So letting $d_2 = 1/c_1 > 0$, $c_2 = 1/d_1 > 0$, we have $d_2 f(x) \leq g(x) \leq c_2 f(x)$ for all positive integers x. Thus, by the definition of Θ with $d = d_2$ and $c = c_2$, $g \in \Theta(f)$.

Ind1 Consider the following program that makes a recursive call to itself:

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Function: Exp(X,Y) ; Input: X, Y Integer;
IF Y=0 THEN Return 1;
      ELSE
      IF (Y MOD 2) =0 THEN Z:=Exp(X , Y DIV 2);Return Z*Z;
                ELSE Return Exp(X,Y-1)*X;
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Prove that for any integer value of $X > 0, Y \geq 0$, $Exp(X, Y) = X^Y$. (Hint: do a proof by strong induction on Y.)

We do the proof by strong induction on the value of Y.

First, for $Y = 0$, $EXP(X, 0) = 1 = X^0$, so the claim is true if $Y = 0$.

Assume that the claim is true for all $0 \leq Y \leq k-1$, i.e. that $EXP(X, Y) = X^Y$ if $0 \leq Y \leq k-1$.

Then if $Y = k$, and k is even, $EXP(X, k) = (EXP(X, k/2))^2 = (X^{k/2})^2 = X^{2(k/2)} = X^k$. (The first equation is from the recursive definition of the program, the second is from the induction assumption for $Y = k/2 \leq k-1$, and the rest is algebra)

If k is odd, then $EXP(X, k) = X(EXP(X, k-1)) = X(X^{k-1}) = X^k$. (The first equation is from the recursive definition of the program, the second is from the induction assumption for $Y = k-1$, and the last is algebra)

So in either case, the claim is also true if $Y = k$.

Thus, by strong induction, the claim is true for all integers $Y \geq 0$.

Ind2 Let S be a set of integers *closed under multiplication*, i.e. if $x \in S$ and $y \in S$, then $xy \in S$. (Some examples are the set of even integers, the set of odd integers, or the set of perfect squares.) Prove that if $x \in S$, then $x^k \in S$ for every $k \geq 1$.

We prove $x^k \in S$ by induction on k. First , we are given $x^1 = x \in S$, so the claim is true for $k = 1$.

Second, assume $x^n \in S$. Since we are given $x \in S$ and S is closed under multiplication, $x^n * x \in S$. So $x^{n+1} \in S$.

Since the claim is true for 1, and if it is true for n it is also true for $n+1$, $x^k \in S$ is true for all integers $k \geq 1$.

Relations Let $S = \{0, 1\}^5$, the set of all 5 element string of 0's and 1's. Consider the relation : $wRw' \iff$ the number of 0's in w = the number of 0's in w'. Is R a partial order, equivalence relation, or neither? If it is a partial order, give a maximal anti-chain. If it is an equivalence relation, how many distinct equivalence classes are there? If neither, give counter-examples that show that R violates the conditions for partial-orders and equivalences.

R is an equivalence relation, since it expresses similarity. It is reflexive, since any w has the same number of 0's as itself, so wRw. If w has the same number of 0's as w', w' has the same number of 0's as w. So $wRw' \to w'Rw$, and R is symmetric. R is transitive, since if wRx and xRy, w has the same number of 0's as x, which also has the same number of 0's as y, so w and y have the same number of 0's, so wRy.

Thus it has all three properties defining an equivalence relation. The equivalence classes will be determined by the possible numbers of 0's in a word with 5 symbols, so there will be one class for each possible number from 0-5, which is six classes total.

Pigeon-Hole Principle Say you knew the following "facts": more than 90% of Americans have one of the 200 most common last names, and more than 90% of Americans have one of the 50 most common first names. Also, at least 80% of Americans are under 40 years old. There are at least 250,000,000 Americans. Prove from these "facts" that there are two Americans with the same (first and last) name who were born on the same day. (Hint: what's the largest number of Americans that there could be with uncommon first names , uncommon last names *OR* that are over 40?)

First, there are at most 10% of Americans with uncommon first names, 10% with uncommon last names and 20% over 40. That means at least 60% of Americans have common first names, common last names and are under 40. 60 % of Americans is at least (3/5) (250,000,000)= 150,000,000. There are 50 common first names and 200 common last names, so at least 150,000,000 /(200)(50) =15,000 of these people have the most common name. On the other hand, there are at most $366 * 40 = 14,640$ birthdays for people under 40. Since there are fewer birthdays than people with the same name, two of the people must have the most common name and the same birthday.