CSE 20, Discrete Mathematics, Winter 1995
Section 1, 1 PM, First Practice Midterm

1. Say that you know the following facts (parenthetical comments are just to make the story coherrent and have no logical importance):

   - To be on time, you must be in New York by 5 PM and your friend must meet you at the airport.
   - To get to New York by 5 PM, you must either connect through Chicago or through Dallas .
   - To connect through Dallas , you must be at the San Diego airport early.
   - If you connect through Chicago, your friend will not meet you at the airport (because the Chicago flight lands at JFK and your friend hates to go to JFK because of the traffic.)

   Prove that to be on time, you must be at the San Diego airport early, starting by formalizing both the premises and conclusion in terms of Boolean algebra.

   Answer: To say that p must happen for q to be true is to say that p is necessary for q or that $q \rightarrow p$. This is somewhat counterintuitive,in that p typically causes q, not the other way. However, what $q \rightarrow p$ means is that : if we know q, we can deduce p, not that q causes p. For example,consider "To succeed you must work hard". If you work hard, you may or may not succeed without being contrary to this maxim. However, if you succeed, it can be deduced from the maxim that you worked hard. With this in mind, let $t \Leftrightarrow$"you are on time", $n \Leftrightarrow$ "you are in New York by 5 PM", $f \Leftrightarrow$ "your friend meets you at the airport, $c \Leftrightarrow$ "you connect through Chicago", $d \Leftrightarrow$ "you connect through Dallas", $s \Leftrightarrow$ "you are at the San Diego airprot early. The premises then become

   - $t \rightarrow (n \wedge f)$
   - $n \rightarrow (c \vee d)$
   - $d \rightarrow e$
   - $c \rightarrow \neg f$

   and what we want to show is $t \rightarrow e$. So in our proof we assume t, and try to prove e.

   (a) $t \rightarrow (n \wedge f)$
   (b) $n \rightarrow (c \vee d)$
   (c) $d \rightarrow e$
   (d) $c \rightarrow \neg f$
   (e) Assume t (assumption, goal now becomes e)
   (f) $n \wedge f$ (5,1, Modus Ponens)
   (g) n (6, Conj. Simp.)
   (h) $c \vee d$ (2, 7, MP)
   (i) f (6, Conj. Simp.)
   (j) $\neg c$ (9, 4, Modus Tollens)
   (k) d (10, 8, Disj. Simp.)
   (l) e (11, 3, MP.)
   (m) Therefore $t \rightarrow e$ (Assumption 1 implies conclusion 12)

2. Let A, B be sets from a common universe. Prove that if $A \cup B = A \cap B$, then $A = B$. : Discussion: Many people made the mistake of getting the GOAL confused with the premises. It is important to find logically equivalent ways of stating the goal, but this should be done ON SEPERATE PAPER, not as part of the proof. Once you have a logically equivalent version of the goal, you prove that logically equivalent version, and then GO BACKWARDS through your reasoning on the seperate paper to show that this indeed proves your goal. VERY IMPORTANT: NEVER ASSUME YOUR GOAL AS PART

OF A PROOF. If you assume your goal to be true, then there is nothing to prove; your argument is completely circular, in that all you are doing is showing that if your goal is true, then your goal is true. Here the goal is to show $A = B$. That is the same as $A \subseteq B \wedge B \subseteq A$. So the proof should have two parts, in the first we show $A \subseteq B$, in the second, vice versa. (Of course, looking at the premises, there is no difference between A and B, so the proof for the second part will be exactly the same as the proof for the first part, swapping A and B. In a les formal proof, we would just say, after proving $A \subseteq B$, "the proof for $B \subseteq A$ is identical". However here we put it in to be complete.) In the first part, then, our goal is to prove $A \subseteq B$ or $\forall x(x \in A \to x \in B)$. Thus, since we are trying to prove a universal rule, we should do a proof by universal generalization. The first step of this part should be to let x_0 be some arbitrary element. Then we want to prove $x_0 \in A \to x_0 \in B$, so our second step should be to assume $x_0 \in A$. Now our goal is to prove $x_0 \in B$. This is not reducible to anything logically simpler, so we start working from what we are given. The premise is that $A \cup B = A \cap B$, so if we can show $x_0 \in A \cup B$ (as really is true, since x_0 in A), we can conclued $x_0 \in A \cap B$, so in particular, $x_0 \in B$, which was our goal. This proof strategy translates to:

(a) $A \cup B = A \cap B$ (premise)

(b) Let x_0 be arbitrary. (No reason needed, this just introduces x_0; we are trying to prove $\forall x(x \in A \to x \in B)$ by universal generalization; by introducing x_0, our goal becomes $x_0 \in A \to x_0 \in B$)

(c) Assume $x_0 \in A$ (assumption, our goal is now to prove $x_0 \in B$)

(d) $x_0 \in A \vee x_0 \in B$ (3, disj. amp.)

(e) $x_0 \in A \cup B$ (4, definition of union)

(f) $x_0 \in A \cap B$ (1,5 substitution)

(g) $x_0 \in A \wedge x_0 \in B$ (6, definiton of intersection)

(h) $x_0 \in B$ (7, Conj. Simp.: this was our sub-goal, so we start working backwards to show that we are actually done)

(i) Therefore, $x_0 \in A \to x_0 \in B$ (assume 3, conclude 8)

(j) Since x_0 was arbitrary, $\forall x(x \in A \to x \in B)$ (2,9, Universal generalization)

(k) $A \subseteq B$ (10, definition of subset; this ends the first part, we now prove $B \subseteq A$ in exactly the same way.)

(l) Let x_1 be arbitrary. (No reason needed, this just introduces x_1; we are trying to prove $\forall x(x \in B \to x \in A)$ by universal generalization; by introducing x_1, our goal becomes $x_1 \in A \to x_1 \in B$)

(m) Assume $x_1 \in B$ (assumption, our goal is now to prove $x_1 \in A$)

(n) $x_1 \in A \vee x_1 \in B$ (13, disj. amp.)

(o) $x_1 \in A \cup B$ (1, definition of union)

(p) $x_1 \in A \cap B$ (1,15 substitution)

(q) $x_1 \in A \wedge x_1 \in B$ (16, definiton of intersection)

(r) $x_1 \in A$ (17, Conj. Simp.: this was our sub-goal, so we start working backwards to show that we are actually done)

(s) Therefore, $x_1 \in B \to x_1 \in A$ (assume 13, conclude 18)

(t) Since x_1 was arbitrary, $\forall x(x \in B \to x \in A)$ (12,19, Universal generalization)

(u) $B \subseteq A$ (20, definition of subset; this ends the second part, we now can conclude $A = B$)

(v) $A = B$ (11, 21, subset criterion for set equality)

3. At a certain party, everyone there shook hands with at least one other person there. Of course, no one shook hands with him/herself. Moreover, for every two people at the party, there was someone they both shook hands with. Formalize these statements using quantifiers (with the universe being the set of people at the party) and the predicate $h(x, y)$ standing for "x shook hands with y". Then formalize and prove from the above that every person at the party shook hands with at least two different people.

Answer: The first premise is a universal rule that applies to every object (person) in our universe. Thus, it should start $\forall x(somethinghappens)$ The something that happens is that x shakes hands with someone, or that there exists someone x shakes hands with. Thus, the first premise is $\forall x \exists y[h(x, y)]$. To say that no-one does something is to either say that there does not exist someone who does that thing, or, equivalently, that everyone does not do the thing. Here the thing is "x shakes own hand" or $h(x, x)$, so we translate the second premise as $\forall x[\neg h(x, x)]$. The third premise is again a general rule, but since it applies to pairs of people rather than individual people, we should start with two variables introduced with universal quantifiers: $\forall x \forall y[something]$. The something that happens is that there is some z that shakes hands with both x and y. Hence, the third premise is $\forall x \forall y[\exists z[h(x, z) \wedge h(y, z)]]$. Our goal is to prove that something happens for every person, so should start $\forall x[something]$. The something that happens is that there exist two different people x shook hands with. Thus, we should next have two new variables introduced with existential quantifiers. $\forall x \exists y \exists z(something)$. What are we told should be true of y and z? First they should both shake hands with x, so $h(x, y)$ and $h(x, z)$. Second, and more subtly, they should be different, i.e., it wouldn't count if $y = z$, so we must have $y \neq z$. Thus our goal is to prove $\forall x[\exists y \exists z[h(x, y) \wedge h(x, z) \wedge y \neq z]]$.

Since our goal is a universal statement, we should use a proof by universal generalization, Introduce x_0 in the first step, and then prove $\exists y \exists z[h(x_0, y) \wedge h(x_0, z) \wedge y \neq z]$. This subgoal is existential so we need to prove it using existence by example. Thus, we need to find values y_0, z_0 for y and z, and then prove $h(x_0, y_0), h(x_0, z_0)$, and $y_0 \neq z_0$. Looking at the premises, the first premise applied to x_0 gives a suitable value for y, and then we can apply the third premise to the pair x_0, y_0 to get z_0. Finally, we can use the second premise to show $y_0 \neq z_0/$

This leads to the following proof:

(a) $\forall x \exists y[h(x, y)]$ (premise)

(b) $\forall x \neg h(x, x)$ (premise)

(c) $\forall x \forall y[\exists z[h(x, z) \wedge h(y, z)]$.(premise)

(d) Let x_0 be an arbitrary person at the party (No reason needed , goal becomes to prove $\exists y \exists z[h(x_0, y) \wedge h(x_0, z) \wedge y \neq z]$.)

(e) $\exists y[h(x_0, y)]$ (Univ. Spec., 1 , $x = x_0$)

(f) Choose y_0 so that $h(x_0, y_0)$ (Choice, 6)

(g) $\exists z[h(x_0, z) \wedge h(y_0, z)]$ (3, U. S. , $x = x_0, y = y_0$)

(h) Choose z_0 so that $h(x_0, z_0) \wedge h(y_0, z_0)$. (Choice, 8; Note that our goal is to prove three things: $h(x_0, z_0), h(x_0, y_0)$ and $y_0 \neq z_0$; only this last hasn't been shown.)

(i) Assume $y_0 = z_0$ (assumption, to prove $y_0 \neq z_0$ by contradiction)

(j) $h(y_0, y_0)$ (9, substitution)

(k) $\neg h(y_0, y_0)$ (2, US $x = y_0$)

(l) From this contradiction, our assumption was incorrect, so $y_0 \neq z_0$ (10,11,12)

(m) $h(x_0, y_0) \wedge h(x_0, z_0) \wedge y_0 \neq z_0$ (7,9, 13)

(n) $\exists y \exists z[(h(x_0, y) \wedge h(x_0, z) \wedge y \neq z]$ (14, existence by example, $y = y_0, z = z_0$)

(o) Since x_0 was arbitrary, $\forall x \exists y \exists z[h(x, y) \wedge h(x, z) \wedge y \neq z]$. (14, Univ. Gen.)