CSE 20, Discrete Mathematics, Winter 1995
Section 1, 1 PM, First Practice Midterm

1. In "The Prince", Machiavelli gives an argument that it is better for a prince to be feared than to be loved. He starts with the following premises (parenthetical comments are just to make the story coherrent, and have no logical content):

   - If the prince is to rule, he must be loved or feared.
   - For the prince to be loved, it is necessary for him to give his supporters favors.
   - If the times are troubled, he cannot give his supporters favors (because his treasury will be emptied by war, plague, or whatever else is causing the troubles.)

   From these premises, he concludes that if the times are troubled, and the prince is to rule, then he must be feared. Show that, (while Machiavelli's morals are questionable), his logic is sound, by formalizing the above premises and conclusion, and giving a proof that the conclusion follows from the premises.

   Answer: First, note that the construction "for p, q must be true" is equivalent to "q is a necessary condition for p" or $p \rightarrow q$. Although p does not cause q, and in general, you could say that q to some extent causes q, if we know p, then we can deduce q, Thus, if we let $r \Leftrightarrow$"The prince rules", $l \Leftrightarrow$ "The prince is loved", $f \Leftrightarrow$ "The prince is feared". $t \Leftrightarrow$ "The times are troubled". and $s \Leftrightarrow$ "The prince gives his supporters favors", the premises translate to:

   - $r \rightarrow (l \vee f)$
   - $l \rightarrow s$
   - $t \rightarrow \neg s$

   and the conclusion is $(t \wedge r) \rightarrow f$. Since our goal is of the form something implies something, we assume the hypothesis and try to prove the conclusion.

   (a) $r \rightarrow (l \vee f)$ (premise)
   (b) $l \rightarrow s$(premise)
   (c) $t \rightarrow \neg s$ (premise)
   (d) Assume $t \wedge r$ (assumption, our goal is now to prove f)
   (e) t (4, Conj. Simp.)
   (f) $\neg s$ (3,4, Modus Ponens)
   (g) $\neg l$ (2,5 Modus Tollens)
   (h) r (4, Conj. Simp.)
   (i) $l \vee f$(1, 7, Modus Ponens)
   (j) f (6,8, Disj. Simp.)
   (k) Therefore, $(t \wedge r) \rightarrow f$ (Assumption 4 implies conclusion 9)

   Let A, B, C be sets from a common universe. Prove that if $\neg(A \subseteq B)$ and $A \subseteq C$, then $\neg(C \subseteq B)$.

   Answer: Discussion: First our goal is to prove $\neg(C \subseteq B)$. This is equvalent to: $\neg\forall x(x \in C \rightarrow x \in B) \Leftrightarrow \exists x[\neg(x \in C \rightarrow x \in B)] \Leftrightarrow \exists x[x \in C \wedge x \notin B]$, so our goal is to find some x that is in C but not in B. Then we can finish the proof by first using "existence by example", and then going backwards through the above equivalences. We are given $\neg(A \subseteq B) \Leftrightarrow \exists x(x \in A \wedge x \notin B)$, so we will use this to find our x, using choice to get an x in A but not in B. We then need to prove that $x \in C$, so we use the other premise $A \subseteq C$.

   (a) Assume $\neg(A \subseteq B)$, and $A \subseteq C$. (assumption, goal is to prove $\neg(C \subseteq B)$.)
   (b) $\neg(\forall x(x \in A \rightarrow x \in B))$. (1, definition of subset)
   (c) $(\exists x(\neg(x \in A \rightarrow x \in B)))$ (2, negation rule for quantifiers)

(d) $\exists x(x \in A \wedge x \notin B)$ (3, negation rule for implication)

(e) Pick x_0 so that $x_0 \in A \wedge x_0 \notin B$ (4, rule of choice)

(f) $\forall x(x \in A \rightarrow x \in C)$(1, definition of subset)

(g) $x_0 \in A \rightarrow x_0 \in C$ (6, Universal spec., $x = x_0$)

(h) $x_0 \in C$ (5,7, Modus ponens)

(i) $x_0 \in C \wedge x_0 \notin B$ (5,8)

(j) $\exists x[x \in C \wedge x \notin B]$ (9, exist. by example)

(k) $\exists x[\neg(x \in C \rightarrow x \in B)]$(10. negation rule for implication)

(l) $\neg\forall x[x \in C \rightarrow x \in B]$ (11, negation rule for quantifiers)

(m) $\neg(C \subseteq B)$ (12, definition of subset)

(n) Therefore, if $\neg(A \subseteq B) \wedge A \subseteq C$, then $\neg(C \subseteq B)$ (Assumption 1 implies concl. 13)

2. A combinatorial optimization problem associates with each possible "solution" S a numerical "value" $v(S)$. A solution S is *optimal* if there is no solution S' with a strictly higher value. Formalize this definition using quantifiers (with the universe taken to be the set of possible solutions), and then formalize and prove that for any two solutions, if both are optimal, then they have the same value.

Answer: The first sentence does not need to be formalized, all it does is tell you what types of objects the variables and expressions represent: S, etc., are "solutions" whereas S's "value",$v(S)$ is a number, and so can be added, compared etc, using normal algebra. The definition is of a predicate that takes one object of type "solution" as input. Thus, the translation should start $opt(S) \Leftrightarrow$ ... Note that, as I said in class, when giving a definition, it is the convention to frequently drop the "only if", but it should be understood. Since what we are defining is a function of S, S should be thought of as a particular solution, and we are defining what it means for S to be optimal. Thus, S will be "free" in the definition, meaning not involved in any quantifier, as opposed to the other variables, which should all be "bound" (quantified over exactly once). The form of the definition is "there is no S' so that.." which should be translated $\neg\exists S'$something. The something is that S' has a strictly larger value than S, or $v(S') > v(S)$. Putting it together we get $opt(S) \Leftrightarrow \neg(\exists S'[v(S') > v(S)])$. Note that S' is bound, but S is free. Our goal is to prove: $\forall S, S'[opt(S) \wedge opt(S') \rightarrow (v(S) = v(S'))]$. Since our goal is a universal rule, we will need to use universal generalization as the last step. Thus, in the first step we pick two arbitrary solutions S_0 and S_1. We then need to show $opt(S_0) \wedge opt(S_1) \rightarrow (v(S_0) = v(S_1))$, so as the second step we assume the former and try to prove that $v(S_0) = v(S_1)$. What we have to work with is that both S_0 and S_1 satisfy the defintion of optimal.

1. Let S_0 and S_1 be arbitrary solutions.

2. Assume $opt(S_0)$ and $opt(S_1)$. (assumption)

3. $\neg(\exists S'[v(S') > v(S_0)])$ (2, definition of opt)

4. $\forall S'[v(S') \leq v(S_0)]$ (3, negation rule for quantifiers)

5. $v(S_1) \leq v(S_0)$ (4, Univ. Spec., $S' = S_1$)

6. $\neg(\exists S'[v(S') > v(S_1)])$ (2, definition of opt)

7. $\forall S'[v(S') \leq v(S_1)]$ (6, negation rule for quantifiers)

8. $v(S_0) \leq v(S_1)$ (7, Univ. Spec., $S' = S_0$)

9. $v(S_0) = v(S_1)$ (5,8, Algebra)

10. Therefore $(opt(S_0) \wedge opt(S_1)) \rightarrow v(S_0) = v(S_1)$(assumption 2 implies conclusion 9)

11. Since S_0, S_1 were arbitrary, $\forall S, S'[(opt(S) \wedge opt(S')) \rightarrow (v(S) = v(S'))]$ (1,10, univ. generalization)