CSE 20, Discrete Mathematics, Fall 1995
Homework 1, Answer Key

Short problems (2 pts each.):

1.1,7 $m \wedge \neg c$ ("but" has a different English connotation than "and" BUT it has the same interpretation in Boolean logic)

1.1, 9 $n \oplus k$ ("but not both" means the "or" is exclusive)

1.1, 10b. $p \wedge \neg q$ (see problem 7)

1.1, 10d $\neg p \wedge q \wedge \neg r$

1.1, 13

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \vee \neg(p \vee q)$
T	T	T	T	F	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	T

1.1, 24

p	q	r	$p \vee q$	$p \wedge r$	$(p \vee q) \vee (p \wedge r)$	$(p \vee q) \wedge r$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	F	T	T
F	T	F	T	F	T	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Not equivalent, if e.g., p and q are true but r is false (second row)

1.2, 27 No, they didn't lie. The "only if" does not imply that meeting their requirements *guarantees* hiring, only that *not* meeting their requirements *guarantees* not being hired.

1.2, 39 If the program is correct, then it does not produce error messages.

1.3,25 This is an example of a hypothetical syllogism ($p \rightarrow q, q \rightarrow r$; therefore $p \rightarrow r$) which is a valid inference rule.

1.3,28 This is the inverse error. The first statement is of the form $p \rightarrow q$, where p is an "or". The second statement is equivalent to $\neg p$ by DeMorgan's law. But knowing $p \rightarrow q$ tells us nothing about whether q is true or false if p

turns out to be false, so we are not entitled to reach any conclusions from the two statements. In fact, if the numbers were 2 and 3, both premises would be true but the conclusion would be false.

1.3,31 This is of the form $p \to r, q \to r$, therefore $(p \vee q) \to r$. This is valid, I would call it a "proof by cases", Epp calls it "dilemma". Both in addition are using a syllogism argument. "Assume $p \vee q$. Then since $p \to r$ and $q \to r$, in either case r. Therefore, r by dilemma /proof by cases. Since assuming $p \vee q$ allowed us to conclude r, $(p \vee q) \to r$."

1.4 , 2 $P = 1, Q = 1 P \wedge Q = 1, \neg Q = 0, R = 1 \vee 0 = 1$

1.4 , 4 $P = 0, Q = 1, R = 0, P \vee Q = 1, P \vee R = 0, S = 1 \wedge 0 = 0$

1.4 , 25 If there are no switches on, the light is off. Turning any one switch turns the light on. Turning any two turns the light on then off. Turning all 3 on turns the light on again. Thus, the light should be on if exactly one of the switches is on or all three are on. This can be written $l = d \oplus p \oplus c$ where d stands for "driver's door is open", p for "passenger's door is open and $c =$"ceiling switch is forwards", and l is "the light is on".

Full Proofs:

- Convert the following to a symbolic form, and give a proof that the assumptions imply the conclusion. Assumptions:

  1. If I do not have my umbrella, then either I left it in class or at home.
  2. If it rained on my way to class, and I left my umbrella at home, then I would be wet.
  3. It rained on my way to class.
  4. I am not wet, and I do not have my umbrella.

  Conclusion: I left my umbrella in class.

- Solution: Let $u =$ "I have my umbrella," $h =$ "I left it at home," $c =$ "I left it in class," $r =$ "It rained on the way to class," and $w =$ "I am wet."

  Then the premises become:

  1. $\neg u \to (c \vee h)$
  2. $(r \wedge h) \to w$
  3. r
  4. $\neg w \wedge \neg u$
     The conclusion is c.

     In the first way of doing the proof, we prove that the assumptions imply the conclusion by using a proof by contradiction.

5. Assume $\neg c$.
6. $\neg u$ (premise 4, Conjunctive Simplification)
7. $c \vee h$ (premise 1, line 6, Modus Ponens)
8. h (assumption 5, line 7, Disjunctive Simplification)
9. $r \wedge h$ (premise 3, line 8, Conjunction)
10. w (premise 2, line 9, Modus Ponens/Detachment)
11. $\neg w$ (premise 4, Conjunctive Simplification)
12. Therefore, our assumption $\neg c$ is incorrect, since lines 10, 11 are a contradiction, so c.

Here is another way of doing the proof directly, without using contradiction.

1. $\neg u \rightarrow (c \vee h)$ (given)
2. $(r \wedge h) \rightarrow w$ (given)
3. r (given)
4. $\neg w \wedge \neg u$ (given)
5. $\neg w$ (premise 4, conj. simp.)
6. $\neg(r \wedge h)$(line 5, premise 2, modus tollens)
7. $\neg r \vee \neg h$ (line 6, De Morgan's Law)
8. $\neg h$ (line 7, premise 3, disj. simp.)
9. $\neg u$ (premise 4, conj. simp.)
10. $c \vee h$ (line 9, premise 1, modus ponens)
11. c (line 10, line 8, disj. simp.)

- Convert the following to a symbolic form, and give a proof that the assumptions imply the conclusion. Assumptions:

  1. If Norma wants to go to her morning meeting, she must wake up early.
  2. If she goes out, she will get home late.
  3. If she gets home late, and wakes up early, she will be working on little sleep.
  4. Norma will not work on little sleep.

  Conclusion: Norma will either not go out, or will miss her morning meeting.

  Answer : Let m = "Norma goes to her meeting," e = ""Norma wakes up early," o = "Norma goes out," l = "Norma gets home late," and s = "Norma works on little sleep."

  The premises translate to:

1. $m \to e$
2. $o \to l$
3. $(l \wedge e) \to s$
4. $\neg s$ Conclusion: $\neg o \vee \neg m$

   I again first show a proof by contradiction.
5. Assume $\neg(\neg o \vee \neg m)$
6. $o \wedge m$ (assumption 5, DeMorgan's Law)
7. m (line 6, Conjunctive Simplification)
8. e (premise 1, line 7, Modus Ponens/Detachment)
9. o (line 6, Conjunctive Simplification)
10. l (premise 2, line 9, Modus Ponens)
11. $l \wedge e$ (line 8, line 10, Conjunction)
12. s (premise 3, line 11, Modus Ponens)
13. Since line 12 and premise 4 are a contradicition, our assumption in 5 is incorrect, so $\neg o \vee \neg m$.

Another way of doing the proof is using a proof by cases.

1. $m \to e$
2. $o \to l$
3. $(l \wedge e) \to s$
4. $\neg s$
5. $\neg(l \wedge e)$ (premise 3 and 4, modus tollens)
6. $\neg l \vee \neg e$ (line 5, De Morgan)
7. Assume $\neg l$
8. $\neg o$ (assumption 7, premise 2, modus tollens)
9. $\neg o \vee \neg m$ (8, disj. addition)
10. Therefore, $\neg l \to (\neg o \vee \neg m)$(Assuming in 7 $lnotl$, we derived $\neg o \vee \neg m$ on line 9)
11. Assume $\neg e$
12. $\neg m$ (assumption 11, premise 1, modus tollens)
13. $\neg o \vee \neg m$ (line 12, disj. add.)
14. Therefore, $\neg e \to (\neg o \vee \neg m)$(Assuming in 11 $lnote$, we derived $\neg o \vee \neg m$ on line 13)
15. $\neg o \vee \neg m$ (6, 10. 14, dilemma)

- Consider the Boolean function with three inputs x, y, z defined by $f(x, y, z) = x \vee [(x \wedge y) \vee ([\neg y] \vee z)]$. First give a diagram of the obvious Boolean circuit for f. Then design a circuit for f using as few gates as you can and prove that your circuit computes f. Hint: Consider the cases $x = T$ and $x = F$, both in the design and in the proof.).

- Answer:

  When $x = T$, $f(x, y, z) = T \vee (\textit{some expression}) \Leftrightarrow T$.

  When $x = F$, $f(x, y, z) = F \vee ((F \wedge y) \vee (\neg y \vee z)) = \neg y \vee z$

  So we claim that $f(x, y, z) = x \vee (\neg y \vee z)$, because there are two ways f could be true : if x is true or else if $y \wedge z$. To make sure we didn't make a mistake in simplifying, we give a proof for our claim. The proof is by the same cases we used to derive the simplification.

  If $x = T$, then both expressions are true by disjunctive addition, so they are equal.

  If $x = F$, both expressions are equivalent to $\neg y \vee z$ (we have already seen this for f, and in this case $x \vee (\neg y \vee z) \Leftrightarrow F \vee (\neg y \vee z) \Leftrightarrow \neg y \vee z$.

  Thus, in both cases $f(x, y, z) = x \vee (y \wedge z)$. Therefore, it is always true. $x \vee (\neg y \vee z)$ can be implemented with a three gate circuit, one negation and then two or's, in the obvious way.

  Alternate method: Using logical equivalences on p. 14 of Epp: $x \vee ((x \wedge y) \vee (\neg y \vee z)) = (x \vee (x \wedge y)) \vee (\neg y \vee z) = x \vee (\neg y \vee z)$. The first step is the commutative law for or, and the second is the little used, but helpful in this problem, absorbtion law $p \vee (p \wedge q) = p$.

- Epp, Section 1.4 (p. 56) , problem 27.

- Let $f(P, Q) = P \wedge (\neg(P \wedge Q))$ be the output of the first circuit, and let $g(P, Q) = P \wedge \neg Q$ be the output of the second. We'll show $f(P, Q) = g(P, Q)$ by cases. If $P = 0$, both functions are 0, since they are both $P \wedge (\textit{something})$. If $P = 1$, $f(P, Q) = 1 \wedge (\neg(1 \wedge Q)) = 1 \wedge (\neg(Q)) = \neg Q$, and $g(P, Q) = 1 \wedge \neg Q = \neg Q$, so they are also equal in this case. Thus, they are equal in both cases, and so always equal.

- Alternatively, we can use the rules on page 14: $f(P, Q) = P \wedge (\neg(P \wedge Q)) = P \wedge (\neg P \vee \neg Q) = (P \wedge \neg P) \vee (P \wedge \neg Q) = 0 \vee (P \wedge \neg Q) = P \wedge \neg Q$. This uses, in order, DeMorgan's Law, Associative Law, Negation Law, and the Universal bound law.

- It is also legal to just give the truthtable for both circuits and to see that they are legal. This corresponds to doing a proof by cases where the cases are all 4 possible values of P and Q.