Filtering in the Frequency Domain

Image Processing

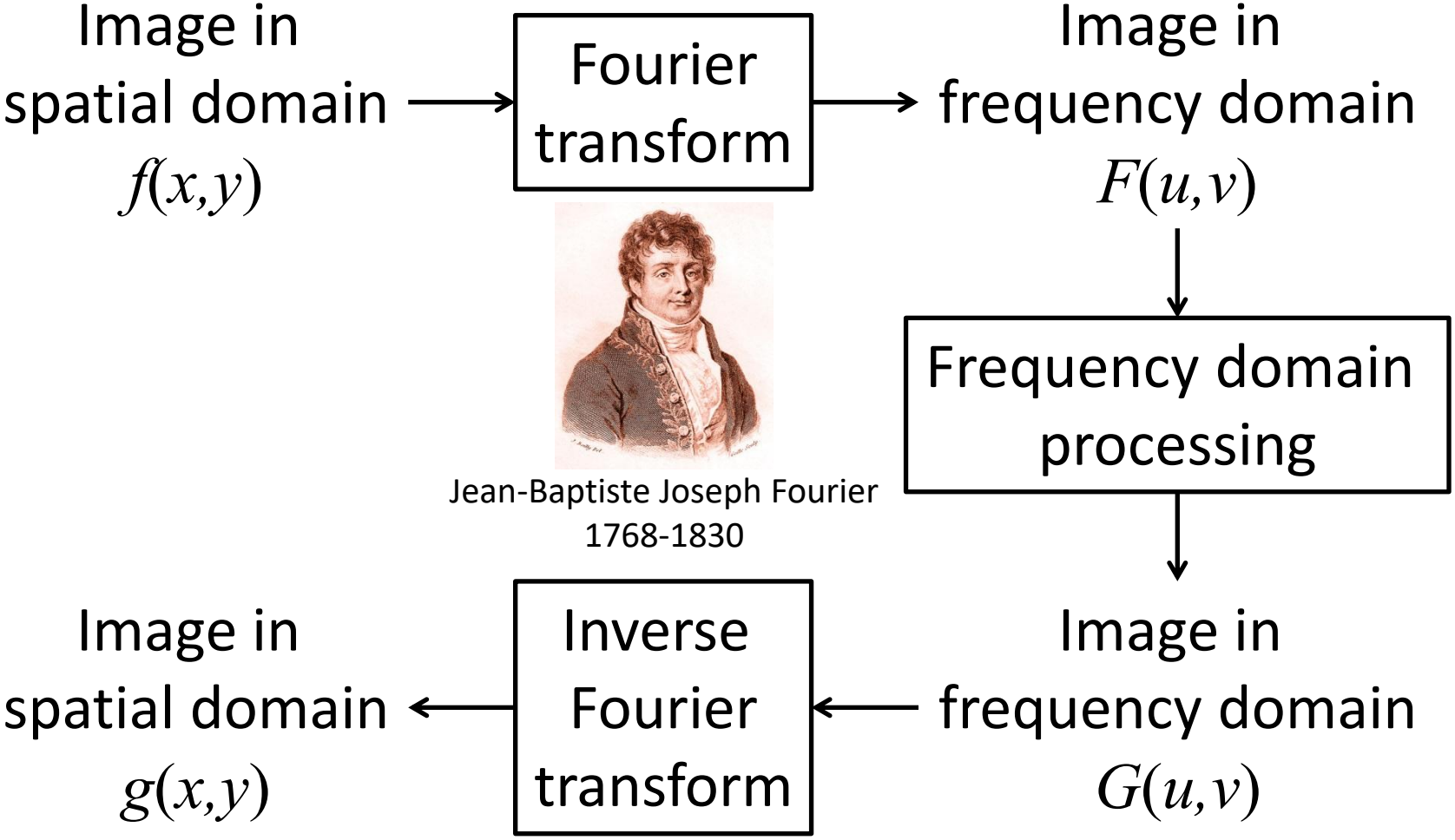
CSE 166

Lecture 7

Announcements

- Assignment 3 will be released today
  - Due Apr 27, 11:59 PM
- Reading
  - Chapter 4: Filtering in the Frequency Domain
    - Sections 4.5, 4.6, 4.7, 4.8, 4.9, and 4.10

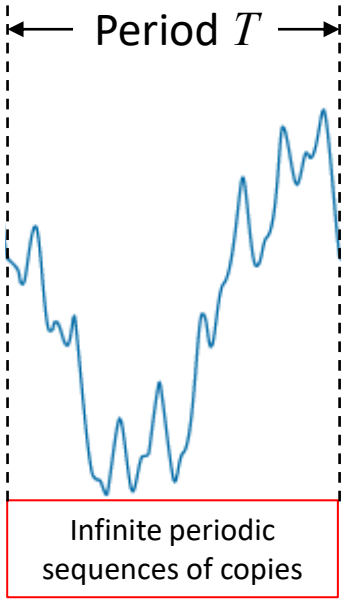
Overview: Image processing in the frequency domain



Jean-Baptiste Joseph Fourier
1768-1830

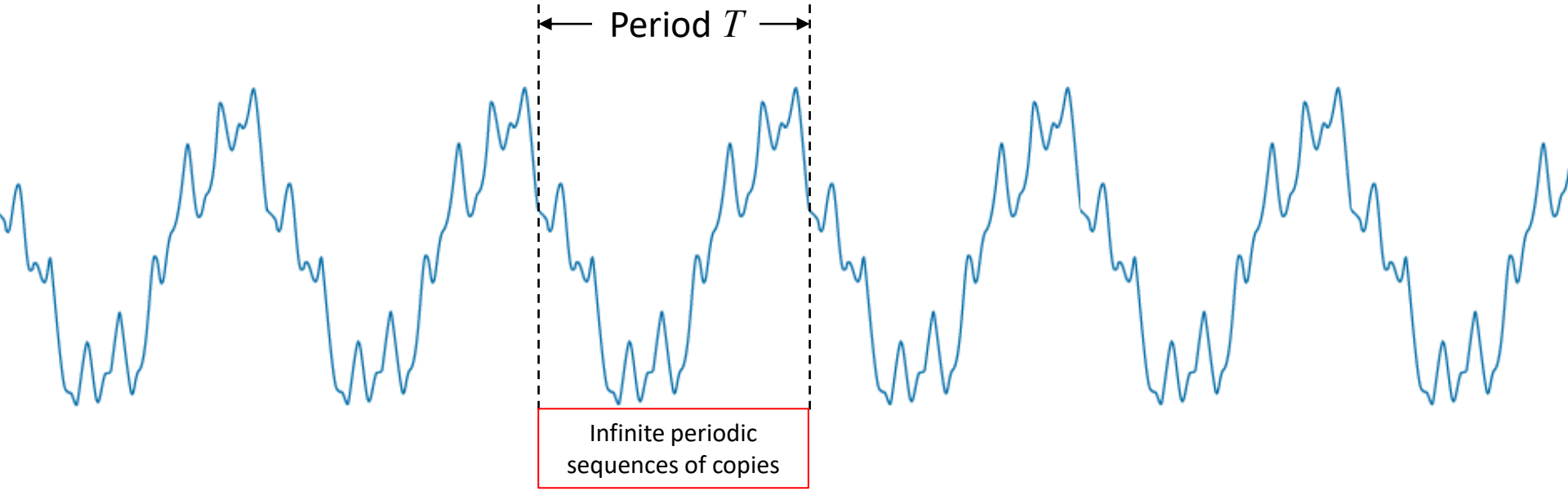
1D periodic functions

- Example periodic function



1D periodic functions

- Example periodic function



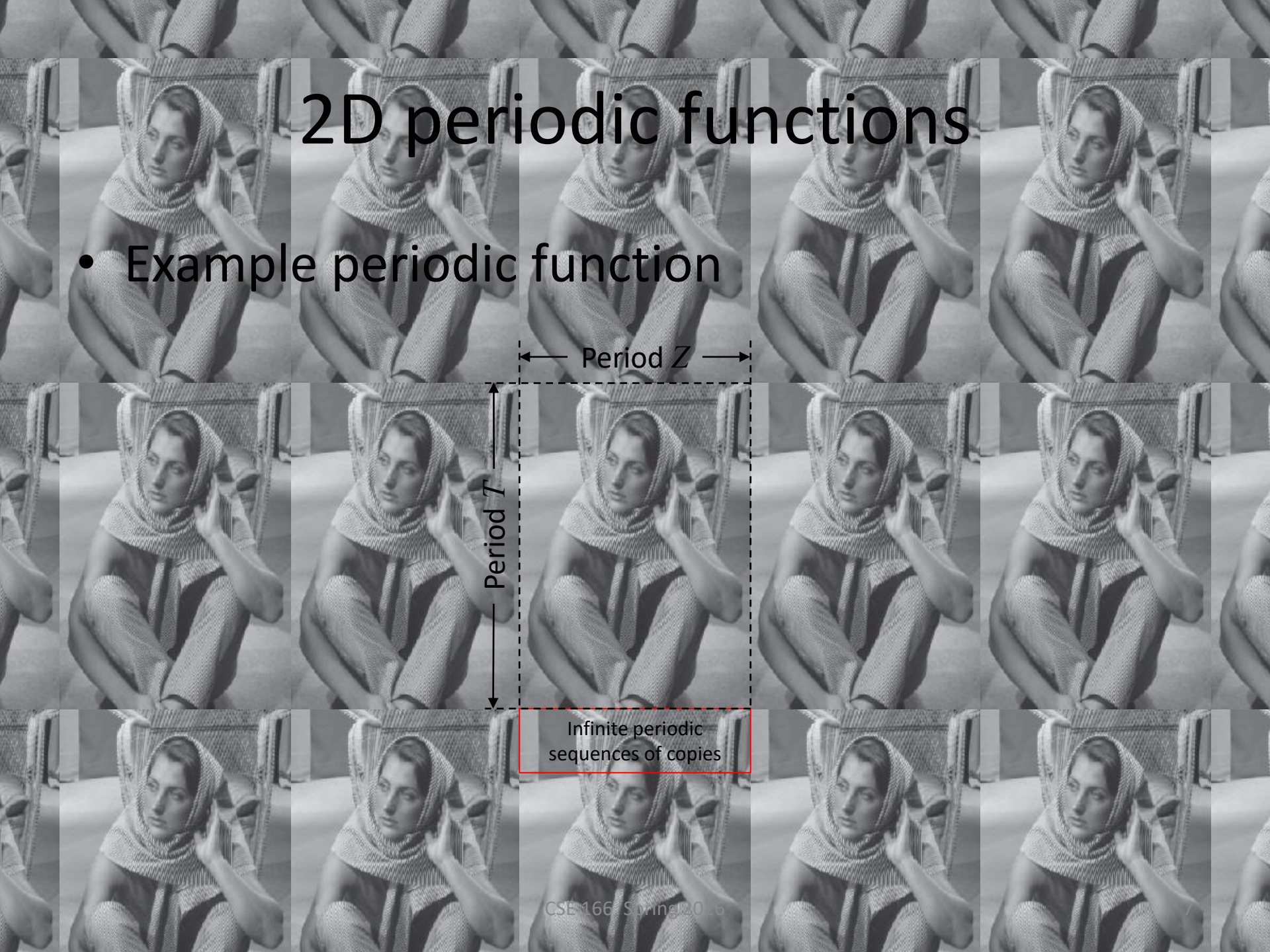
2D periodic functions

- Example periodic function



2D periodic functions

- Example periodic function



1D continuous Fourier transform

- (Forward) Fourier transform

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} \, \mathrm{d}t$$

- Inverse Fourier transform

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} \, \mathrm{d}\mu$$

2D continuous Fourier transform

- (Forward) Fourier transform

$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}\,\mathrm{d}t\,\mathrm{d}z$$

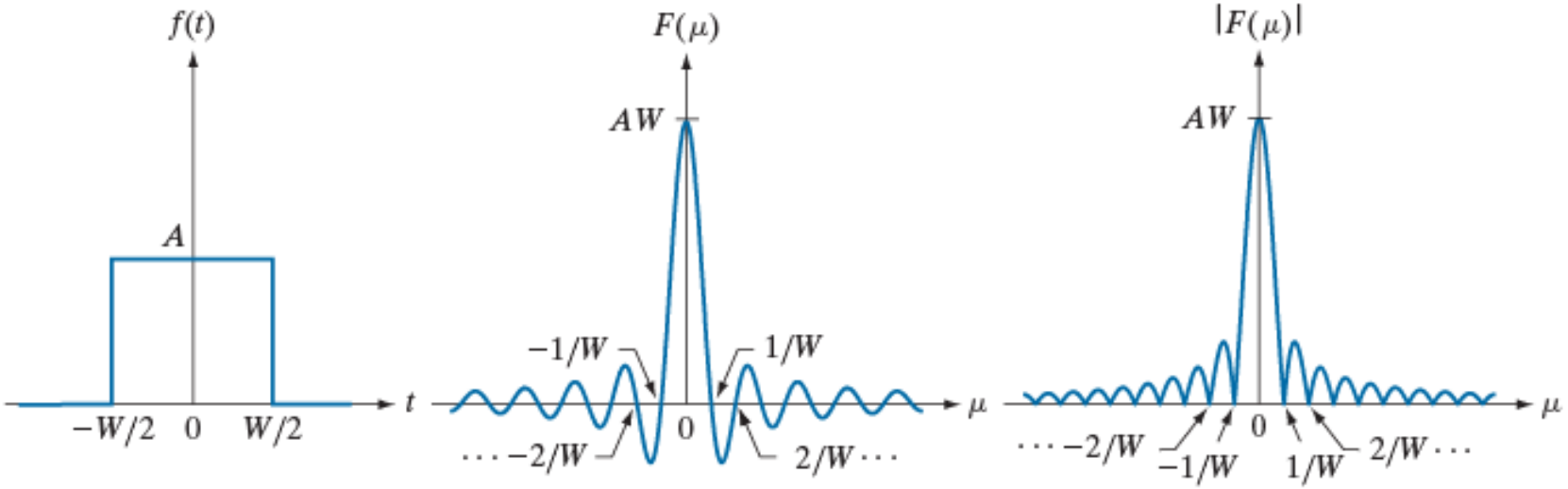
- Inverse Fourier transform

$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu)e^{j2\pi(\mu t+\nu z)}\,\mathrm{d}\mu\,\mathrm{d}\nu$$

2D continuous Fourier transform

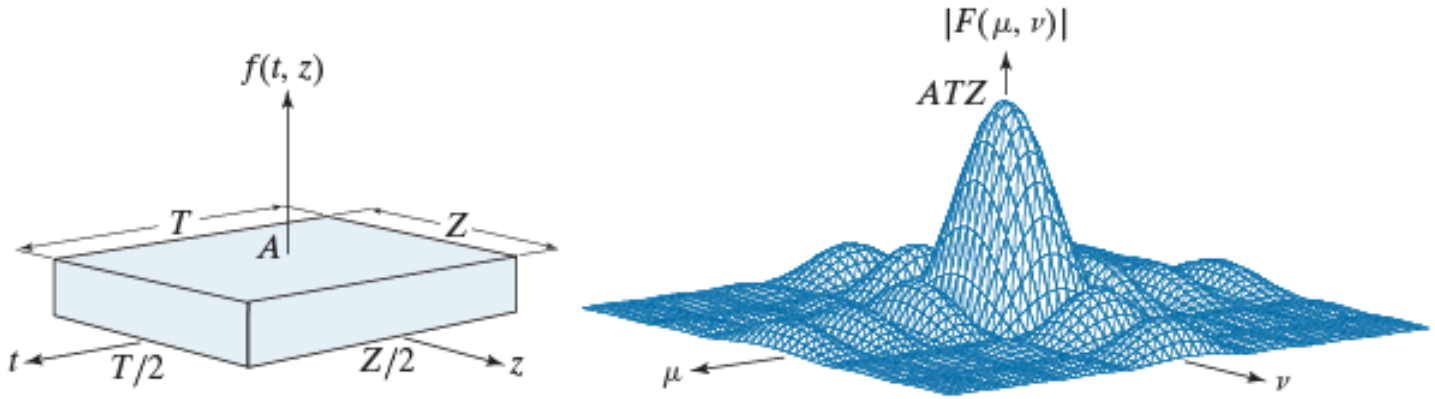
Example: box function

1D



The Fourier transform of a box function does not have an imaginary component (i.e., it is real)

2D



1D impulse function

- Discrete

Discrete variable x (at $x = 0$)

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \left(\text{note } \sum_{x=-\infty}^{\infty} \delta(x) = 1 \right)$$

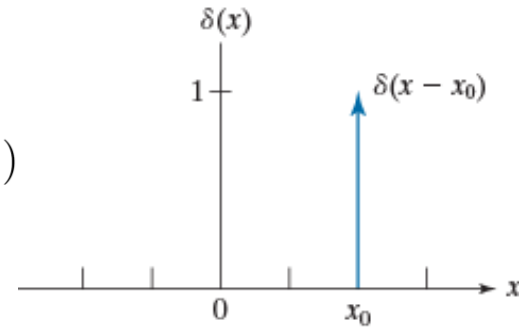
Unit impulse

If $f(x)$ is discrete at $x = 0$, then

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

Generalizes to an arbitrary point x_0

Shifting property $$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$



The value of the function f at the location of the impulse (i.e., sample of the function)

2D impulse function

- Discrete

Discrete variables x and y (at $x = y = 0$)

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \left(\text{note } \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} \delta(x,y) = 1 \right)$$

Unit impulse

If $f(x,y)$ is discrete at $x = y = 0$, then

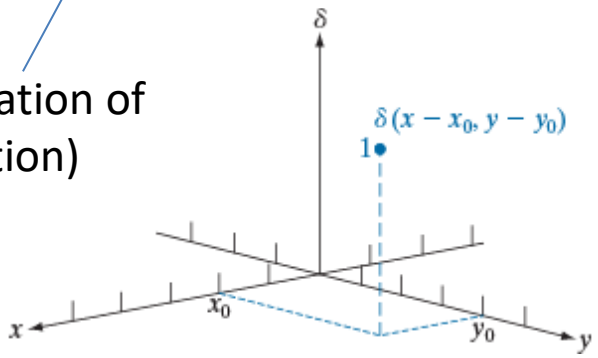
$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y) = f(0,0)$$

Generalizes to an arbitrary point (x_0, y_0)

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0, y-y_0) = f(x_0,y_0)$$

Shifting property

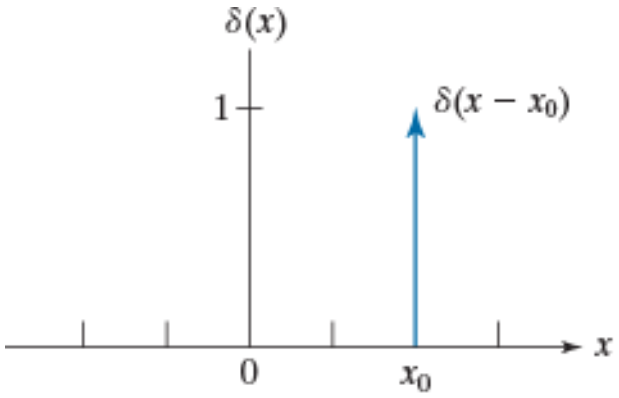
The value of the function f at the location of the impulse (i.e., sample of the function)



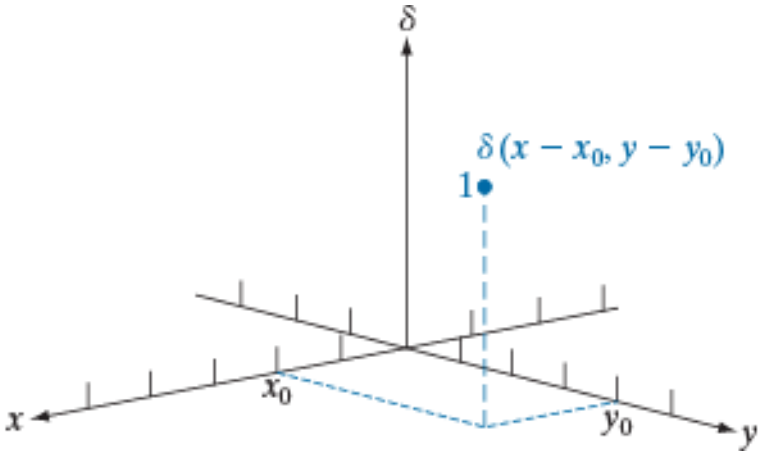
Discrete impulse function

Impulse function

1D



2D

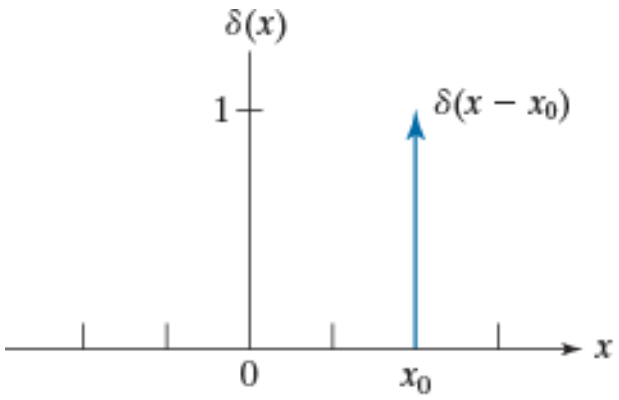


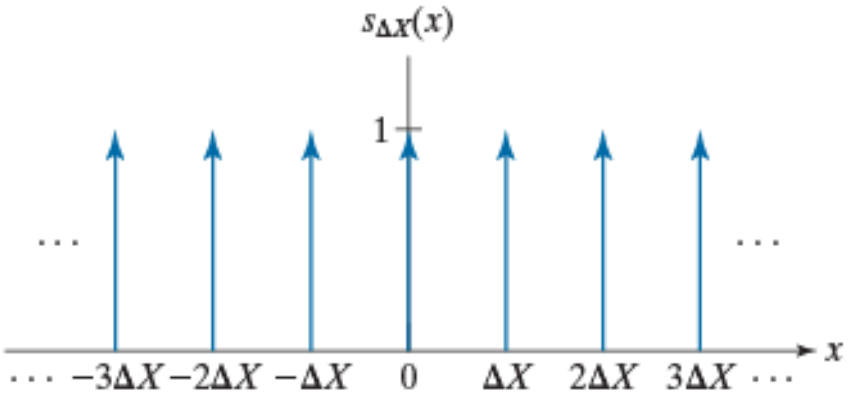
Discrete impulse function and impulse train

Impulse function

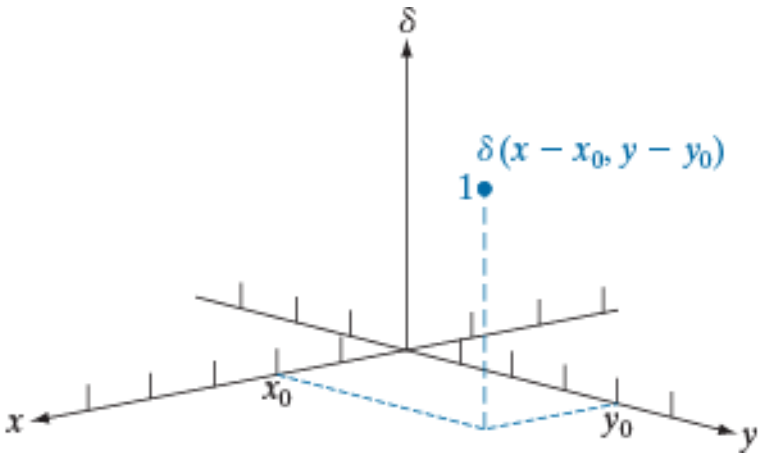
Impulse train

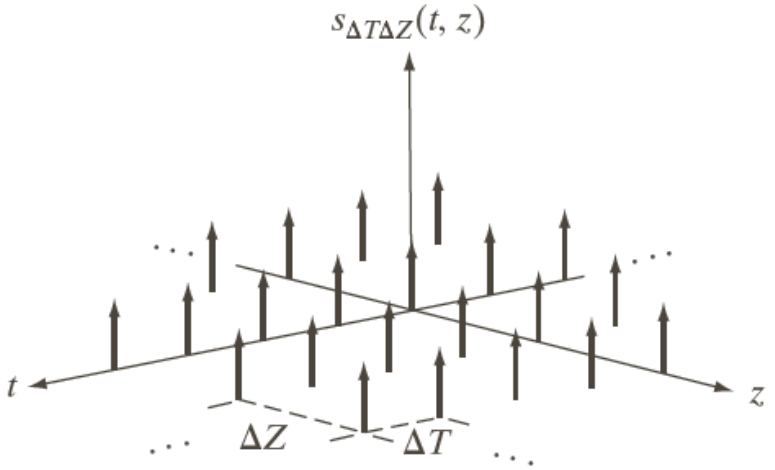
1D





2D

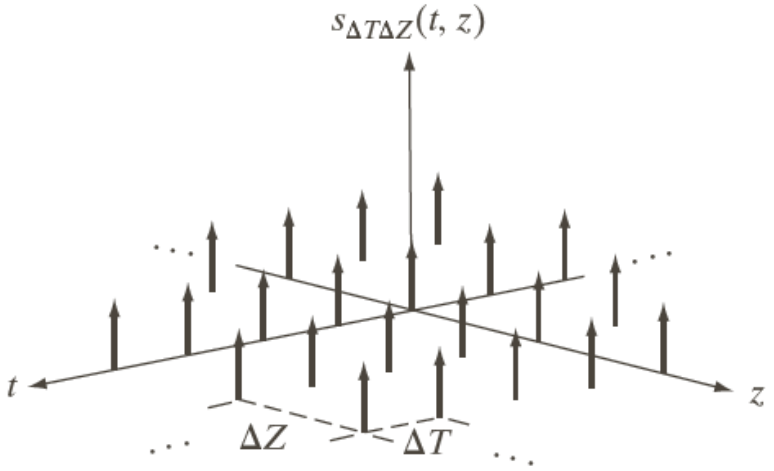




2D discrete impulse train

Sum of infinitely many periodic impulses $\Delta T, \Delta Z$ units apart

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$



$f(t, z)$ can be recovered without error if

$$\Delta T < \frac{1}{2\mu_{\max}} \text{ and } \Delta Z < \frac{1}{2\nu_{\max}}$$

$$\frac{1}{\Delta T} > 2\mu_{\max} \text{ and } \frac{1}{\Delta Z} > 2\nu_{\max}$$

Guaranteed sufficient separation between copies

Sampling

- Sampled function

Continuous function $f(t)$
Sampled function

$$\tilde{f}(t) = f(t)s_{\Delta T}(t)$$

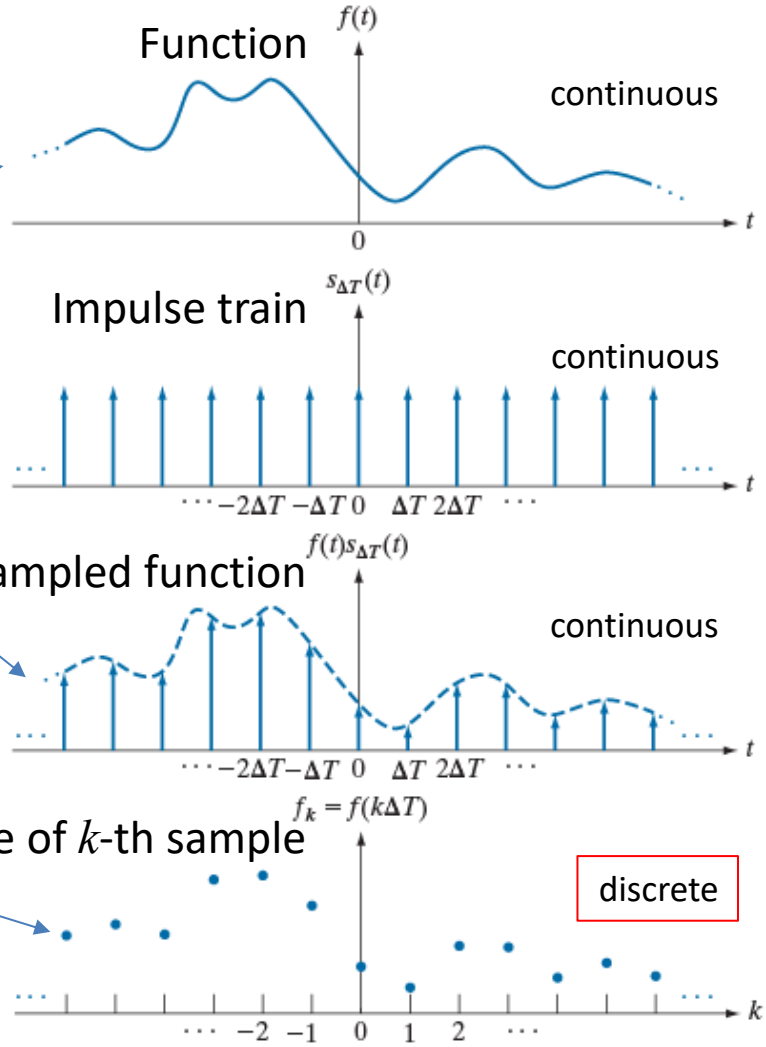
$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

Value of the k-th sample

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\,dt$$

$$f_k = f(k\Delta T)$$

where $k = \ldots, -2, -1, 0, 1, 2, \ldots$



Sampling

- Fourier transform of sampled function

$$\tilde{F}(\mu) = \mathfrak{F}\left\{\tilde{f}(t)\right\}$$
$$\tilde{F}(\mu) = \mathfrak{F}\left\{f(t)s_{\Delta T}(t)\right\}$$
$$\tilde{F}(\mu) = F(\mu)\bigstar S(\mu)$$

Convolution theorem

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu-\tau)\,\mathrm{d}\tau$$
$$\tilde{F}(\mu) = \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta\left(\mu-\tau-\frac{n}{\Delta T}\right)\mathrm{d}\tau$$
$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)\delta\left(\mu-\tau-\frac{n}{\Delta T}\right)\mathrm{d}\tau$$

Shifting property

$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu-\frac{n}{\Delta t}\right)$$

This is an infinite, periodic sequence of copies of $F(\mu)$

Since $F(\mu)$ is continuous, $\tilde{F}(\mu)$ is continuous

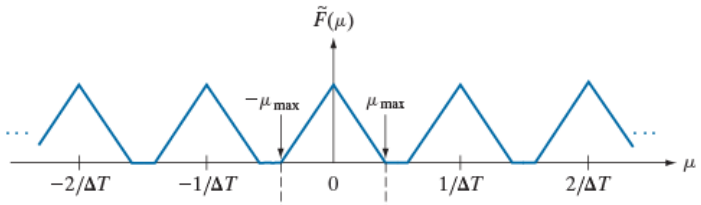
Sampling

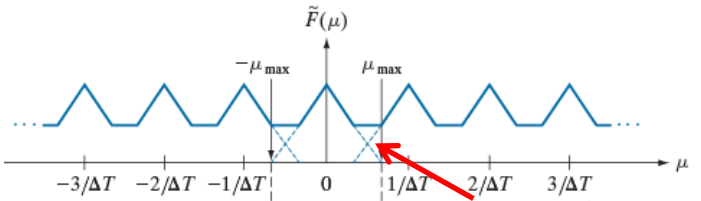
Over-sampled

Under-sampled

Fourier transform of **sampled** function

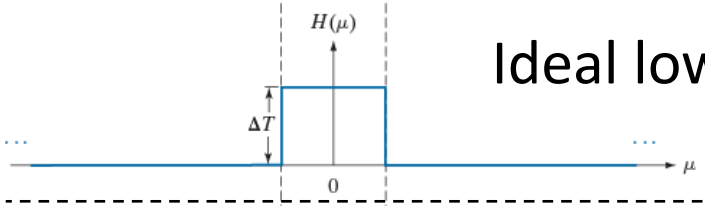
This is an infinite, periodic sequence of copies of $F(\mu)$

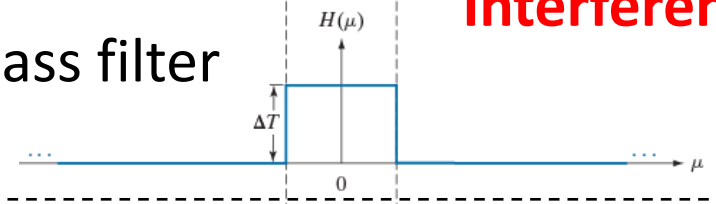




Interference

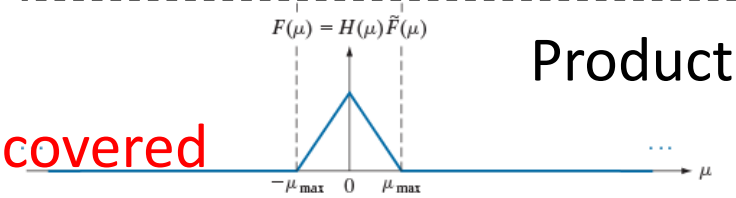
Ideal lowpass filter

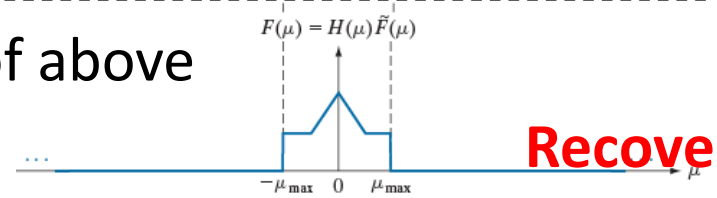




1D

Product of above





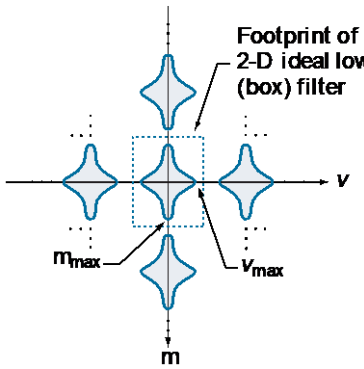
Recovered without error

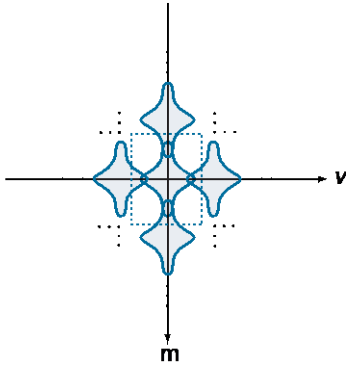
Single copy

Not a single copy

Recovered with error

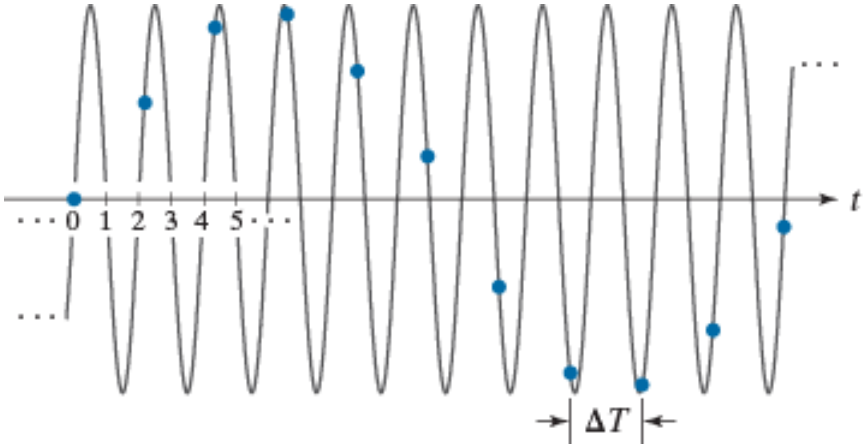
2D





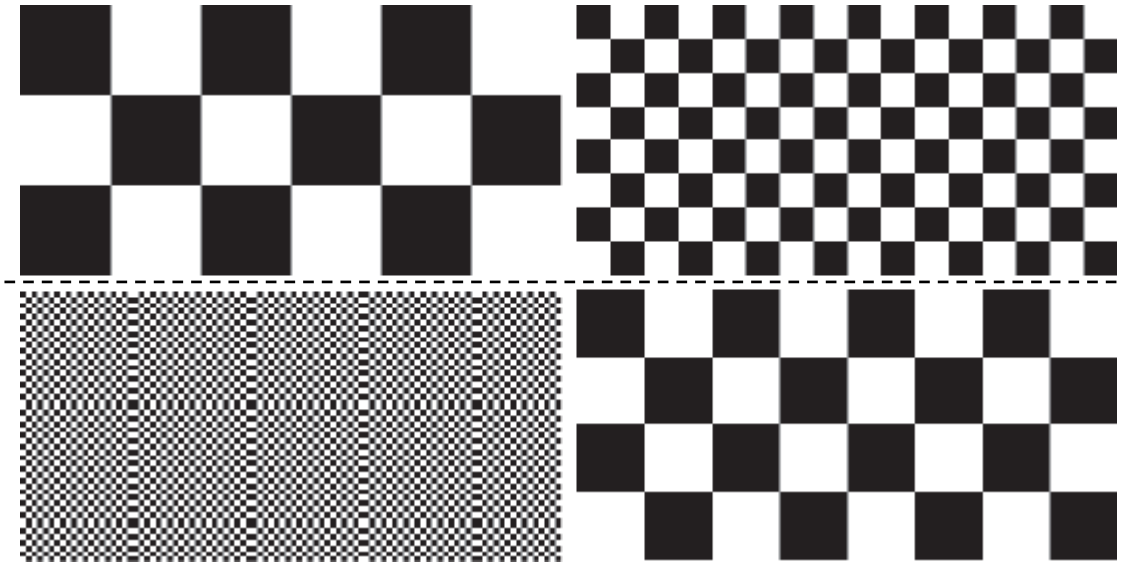
Under-sampling results in aliasing

1D



Under-sampled function looks like a lower frequency sine wave

2D



Original

Aliasing

In general, scaling (zooming or shrinking) an image causes aliasing

Aliasing in real images

Original | Subsample **Aliasing** | Blur then subsample No aliasing



a b c

FIGURE 4.19 Illustration of aliasing on resampled natural images. (a) A digital image of size 772 × 548 pixels with visually negligible aliasing. (b) Result of resizing the image to 33% of its original size by pixel deletion and then restoring it to its original size by pixel replication. Aliasing is clearly visible. (c) Result of blurring the image in (a) with an averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

1D discrete Fourier transform (DFT)

Each of these are an infinite, periodic sequence of copies

- (Forward) Fourier transform

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad u = 0, 1, 2, \ldots, M-1$$

Units of cycles/sample

- Inverse Fourier transform

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \quad x = 0, 1, 2, \ldots, M-1$$

2D discrete Fourier transform (DFT)

Each of these are an infinite, periodic sequence of copies

- (Forward) Fourier transform

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

Units of cycles/sample

$$u = 0, 1, 2, \ldots, M-1 \text{ and } v = 0, 1, 2, \ldots, N-1$$

- Inverse Fourier transform

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

$$x = 0, 1, 2, \ldots, M-1 \text{ and } y = 0, 1, 2, \ldots, N-1$$

2D discrete Fourier transform (DFT)

- Fourier transform pairs

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} \qquad u = 0,1,2,\ldots,M-1 \text{ and } v = 0,1,2,\ldots,N-1$$

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)} \qquad x = 0,1,2,\ldots,M-1 \text{ and } y = 0,1,2,\ldots,N-1$$

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} \qquad u = 0,1,2,\ldots,M-1 \text{ and } v = 0,1,2,\ldots,N-1$$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)} \qquad x = 0,1,2,\ldots,M-1 \text{ and } y = 0,1,2,\ldots,N-1$$

$$F(u,v) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} \qquad u = 0,1,2,\ldots,M-1 \text{ and } v = 0,1,2,\ldots,N-1$$

$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)} \qquad x = 0,1,2,\ldots,M-1 \text{ and } y = 0,1,2,\ldots,N-1$$

2D discrete Fourier transform (DFT)

- Zero-frequency term (DC term)

Note zero-frequency term is propotional to $\bar{f}(x,y)$, the average value of $f(x,y)$

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$F(0,0) = MN\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

$$F(0,0) = MN\bar{f}(x,y)$$

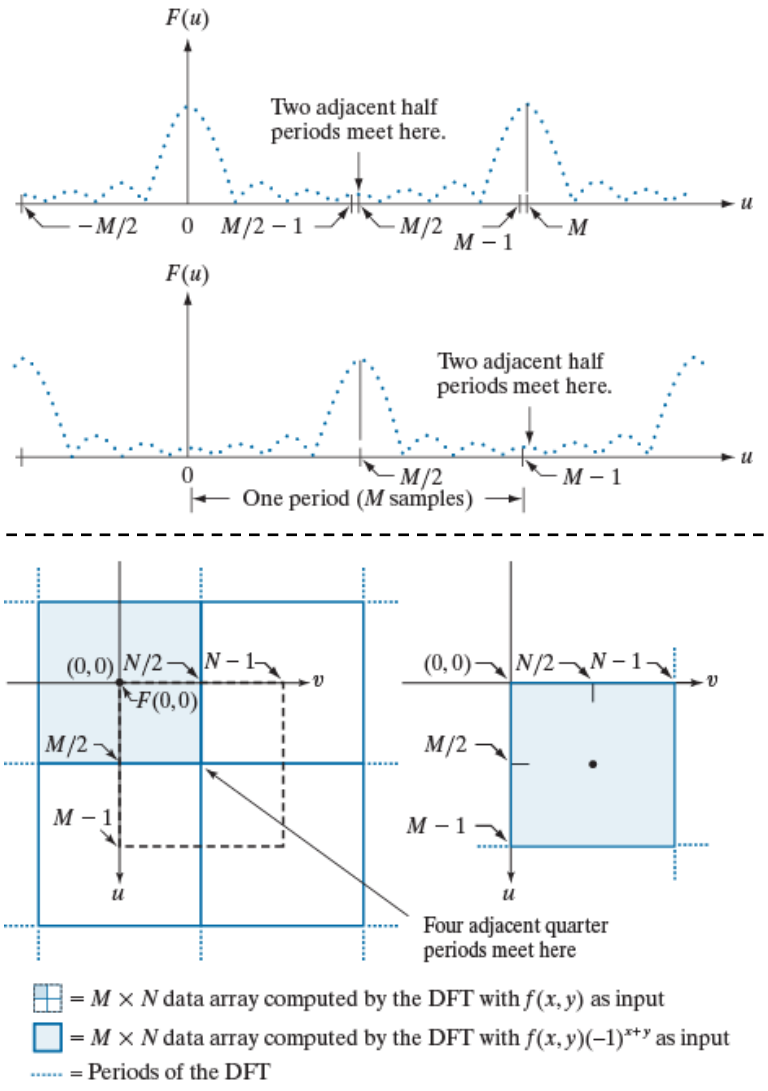
$$\|F(0,0)\| = MN\|\bar{f}(x,y)\|$$

Centering the DFT

1D

Each of these are an infinite, periodic sequence of copies

2D

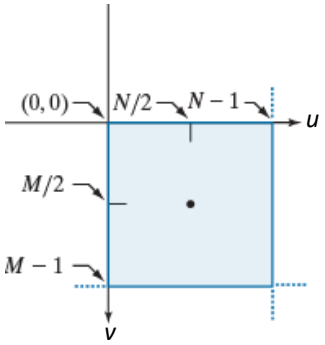


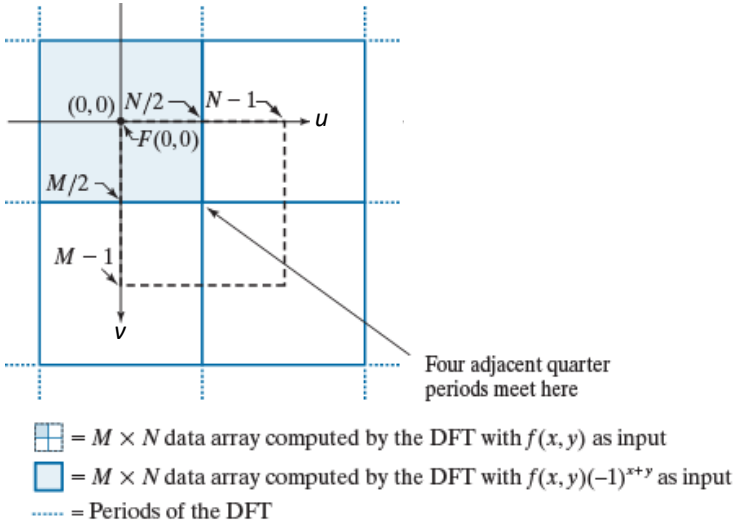
a
b
c d

FIGURE 4.22 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The area within the dashed rectangle is the data array, $F(u,v)$, obtained with Eq. (4-67) with an image $f(x,y)$ as the input. This array consists of four quarter periods. (d) Shifted array obtained by multiplying $f(x,y)$ by $(-1)^{x+y}$ before computing $F(u,v)$. The data now contains one complete, centered period, as in (b).

Centering the DFT

Each of these are an infinite, periodic sequence of copies



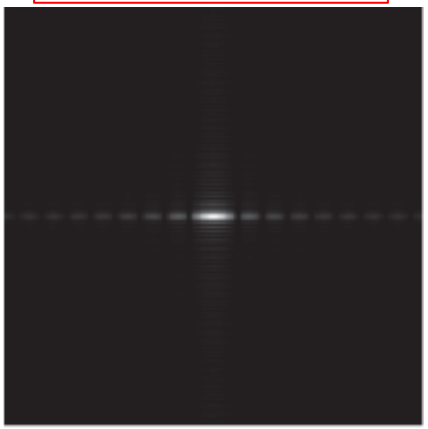


Origin at top left



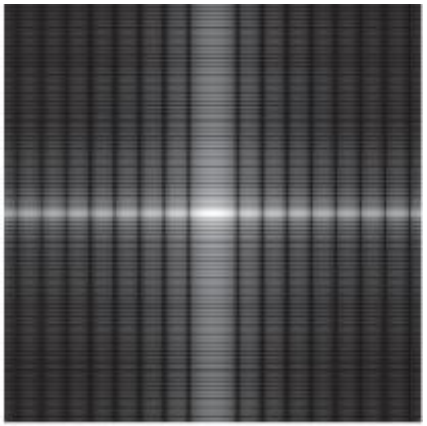
DFT magnitude
(look at corners)

Origin at center



Shifted
DFT magnitude

Origin at center



Log of shifted
DFT magnitude

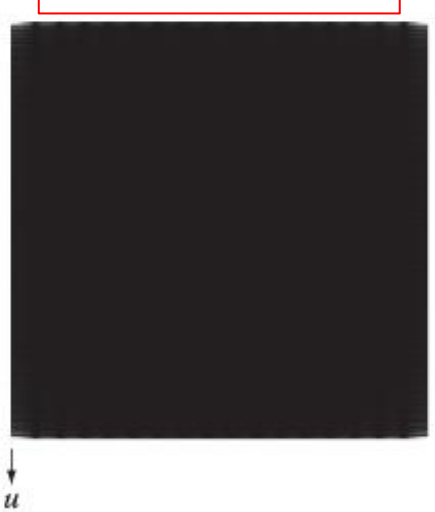
Centering the DFT

Each of these are an infinite, periodic sequence of copies

Origin at top left

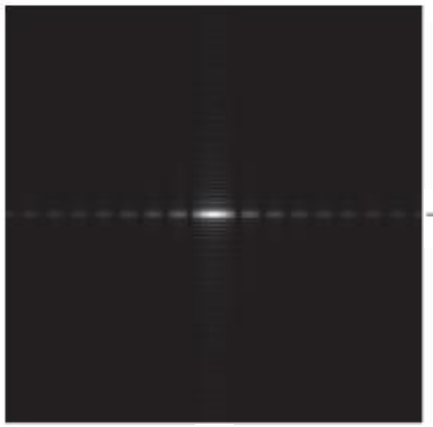
Original

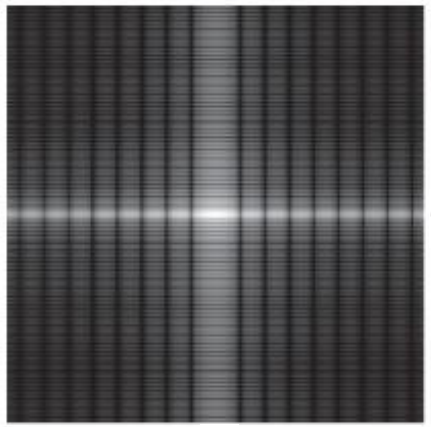




DFT magnitude
look at corners)

Shifted
DFT magnitude





Log of shifted
DFT magnitude

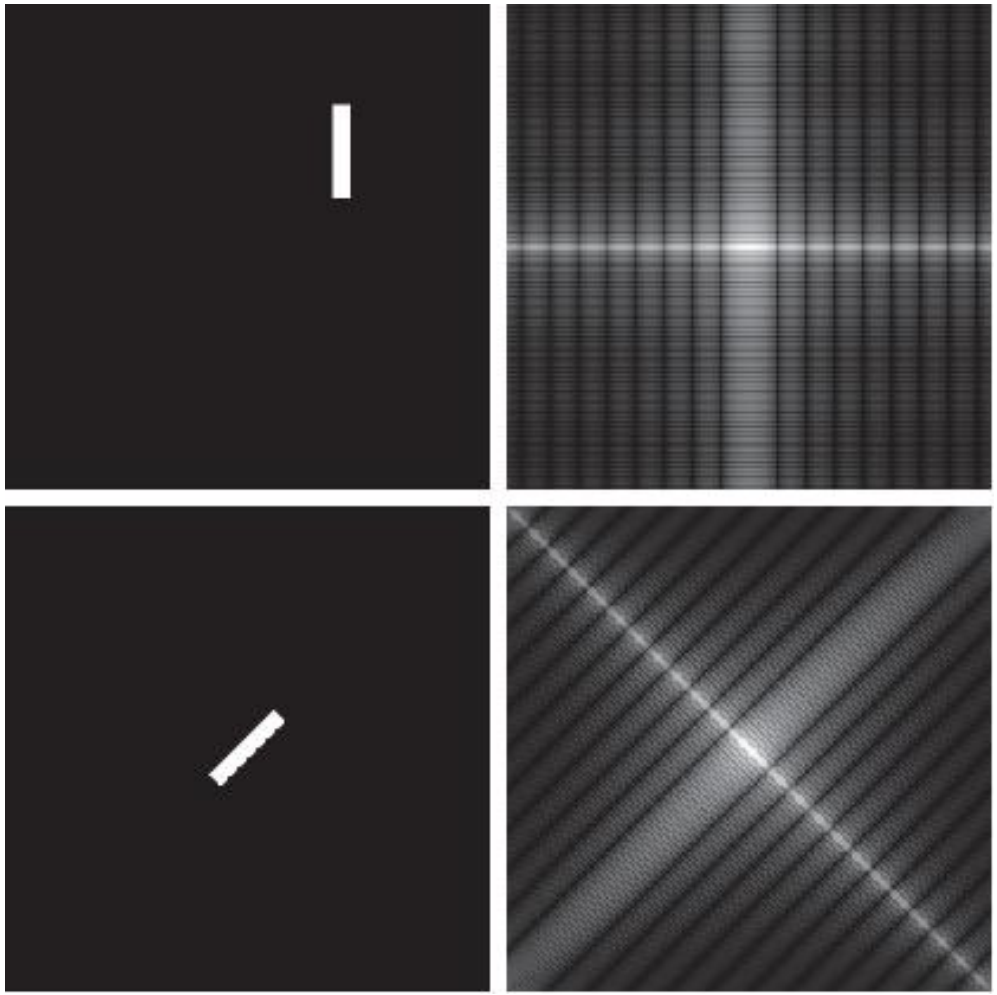
Origin at center

DFT magnitude of geometrically transformed images

Each of these are an infinite, periodic sequence of copies

Translated

Rotated about center



Same magnitude as original (invariant to translation)

Different magnitude as original (not invariant to rotation)

DFT phase of geometrically transformed images

Each of these are an infinite, periodic sequence of copies

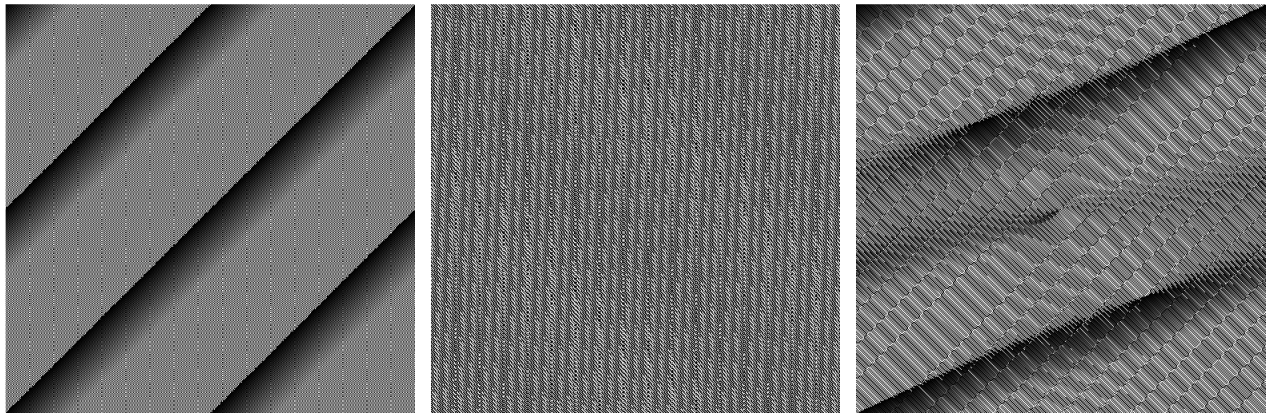
Original

Translated

Rotated about center



DFT phase



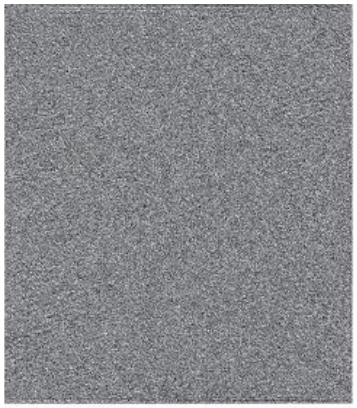
Contributions of magnitude and phase to image formation

Each of these are an infinite, periodic sequence of copies

DFT Phase

Input image







Inverse DFT: Phase only (zero magnitude)

Inverse DFT: Magnitude only (zero phase)







Inverse DFT: Boy magnitude and rectangle phase

Inverse DFT: Rectangle magnitude and boy phase

2D convolution

- 2D discrete convolution

$$f(x,y) \star h(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n)$$

$$x = 0, 1, 2, \ldots, M-1 \text{ and } y = 0, 1, 2, \ldots, N-1$$

- 2D convolution theorem

$$f(x,y) \star h(x,y) \Longleftrightarrow F(u,v) H(u,v)$$

$$f(x,y) h(x,y) \Longleftrightarrow F(u,v) \star H(u,v)$$

Key to filtering in the frequency domain

Circular convolution

- Because the DFT is an infinite, periodic sequence of copies, the convolution is **circular**

$$f(t) \bigstar h(t) \Longleftrightarrow H(\mu)F(\mu)$$

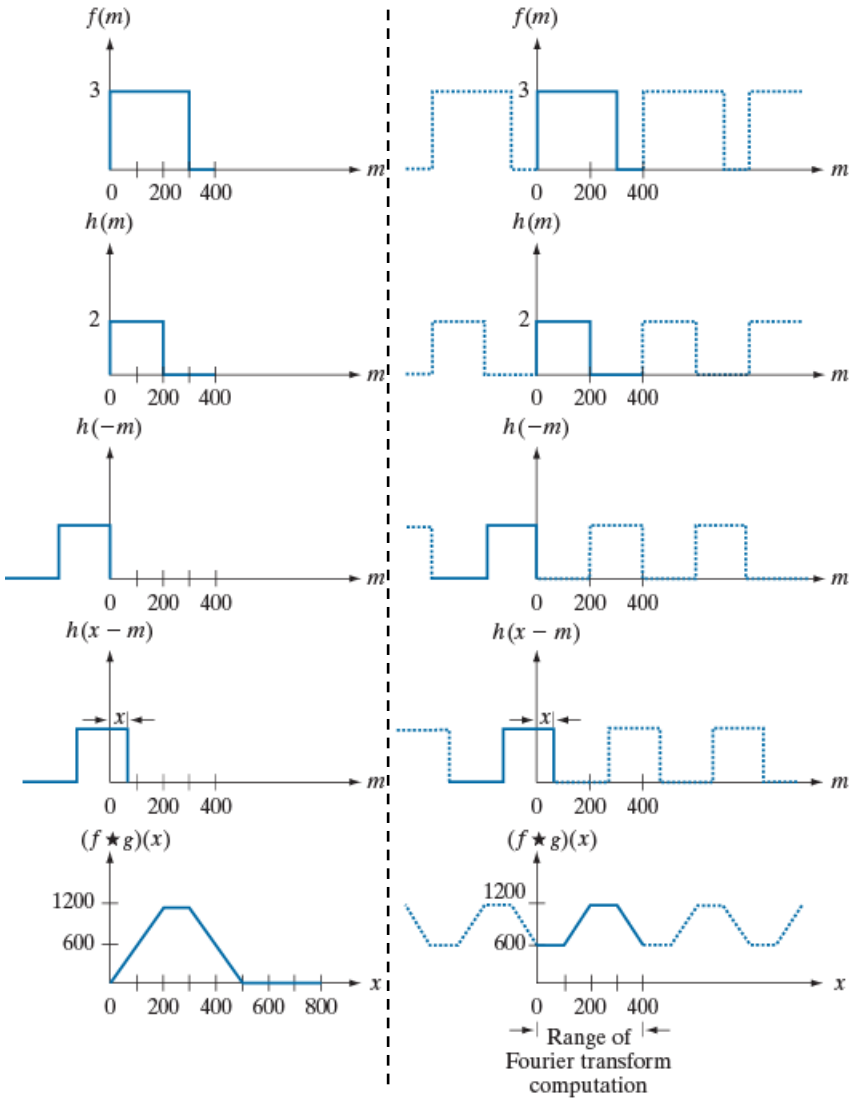
$$f(x,y) \bigstar h(x,y) \Longleftrightarrow F(u,v)H(u,v)$$

Key to filtering in the frequency domain

Circular convolution

Each of these are an infinite, periodic sequence of copies

Filtering in spatial domain using convolution



Filtering in frequency domain using product **without zero-padding**

expected result

wraparound error

Filtering using convolution theorem

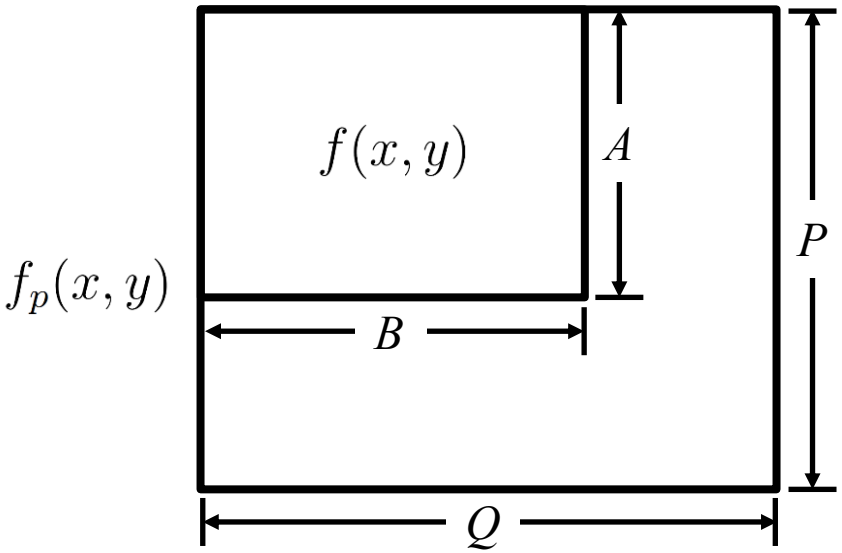
- To avoid wraparound error, both functions must be zero padded
- 1D $f(t) \star h(t) \Longleftrightarrow H(\mu)F(\mu)$

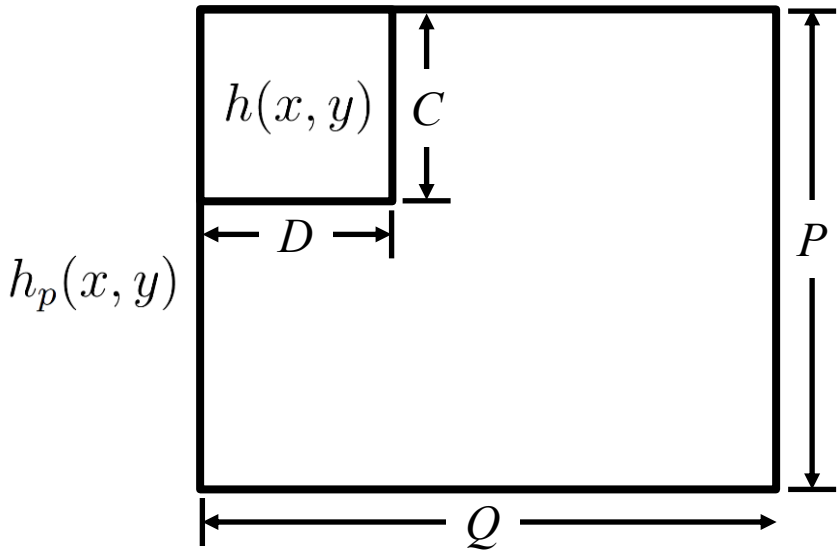
If size of $f(x)$ is A samples and size of $h(x)$ is B samples, then both must be zero padded to size $P \geq A + B - 1$ prior to computing DFTs to avoid wraparound error

Filtering using convolution theorem

- To avoid wraparound error, both functions must be zero padded
- 2D $f(x,y) \star h(x,y) \Longleftrightarrow F(u,v)H(u,v)$

If $f(x,y)$ is $A \times B$ and $h(x,y)$ is $C \times D$, then both must be zero padded to $P \times Q$, where $P \geq A + C - 1$ and $Q \geq B + D - 1$

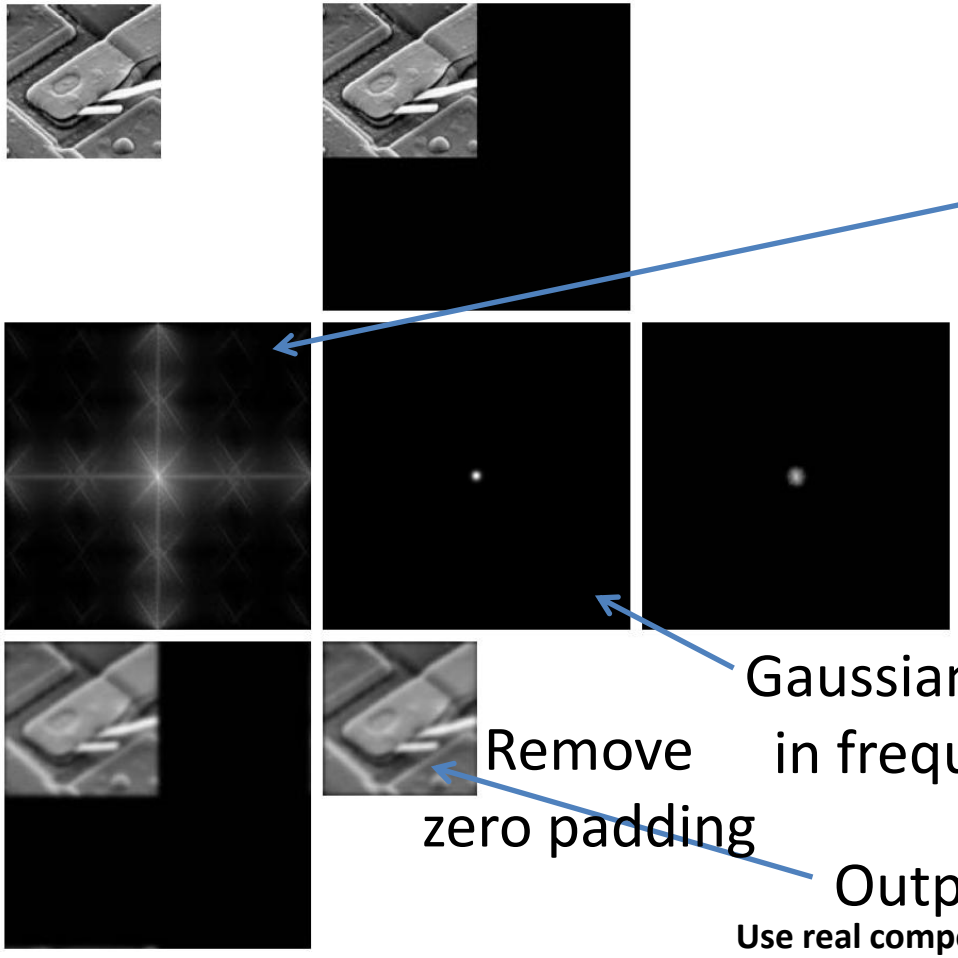




Filtering using convolution theorem

Input image

Zero padding



Fourier transform, then center

Product in frequency domain

Gaussian lowpass filter in frequency domain

Remove zero padding

Output image

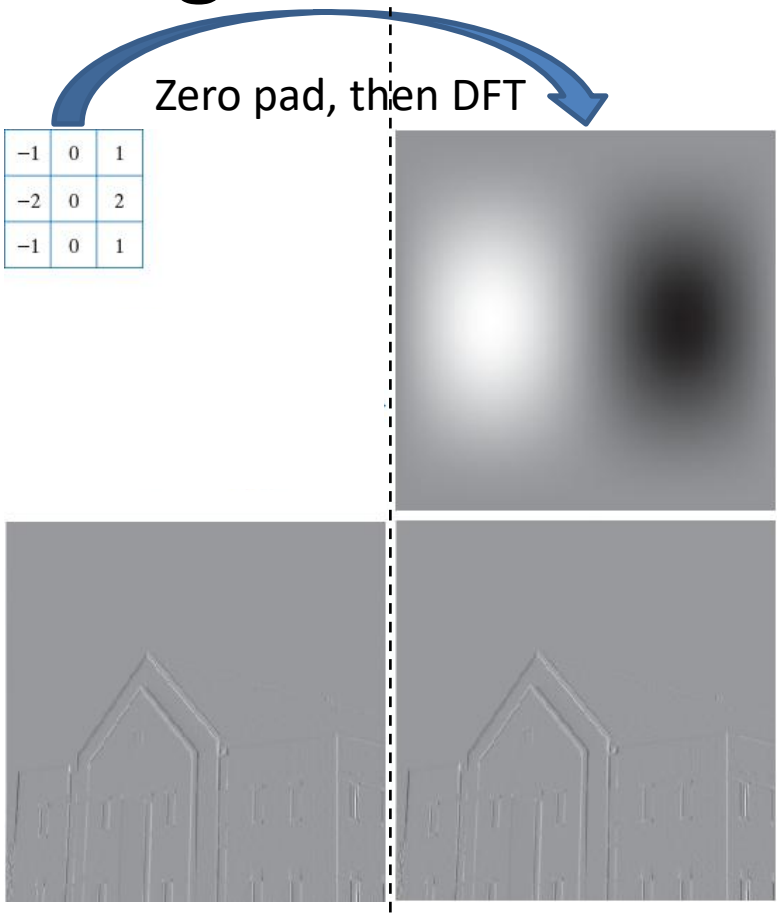
Inverse center, then inverse Fourier transform

Filtering in frequency domain using product **with zero-padding**

no wraparound error

Use real component only.
Mathematically, imaginary component will be 0, but due to floating point errors, it may be slightly nonzero.

Filtering using convolution theorem



Filtering in spatial domain using convolution

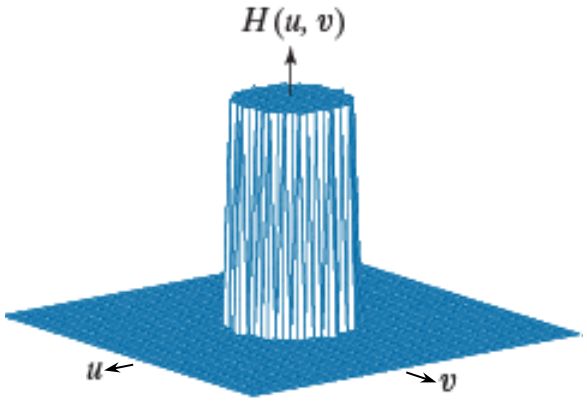
Filtering in frequency domain using product

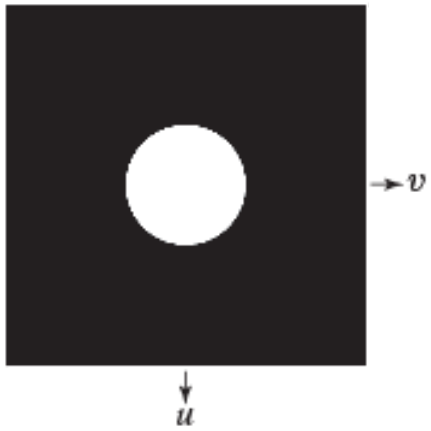
$$f(x,y) \bigstar h(x,y) = \mathfrak{F}^{-1}\{F(u,v)H(u,v)\}$$

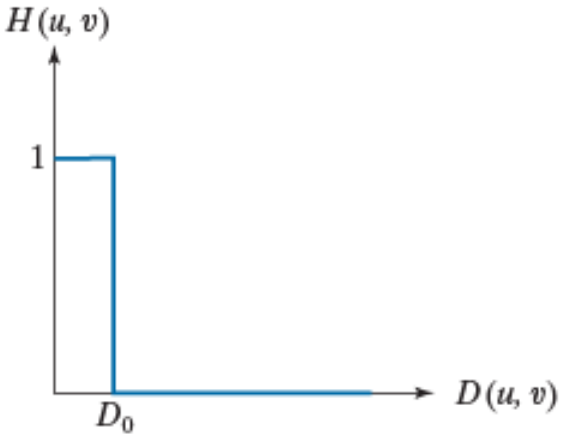
Identical results

Filtering in the frequency domain

- Ideal lowpass filter (LPF)
  - Filter must be constructed in frequency domain







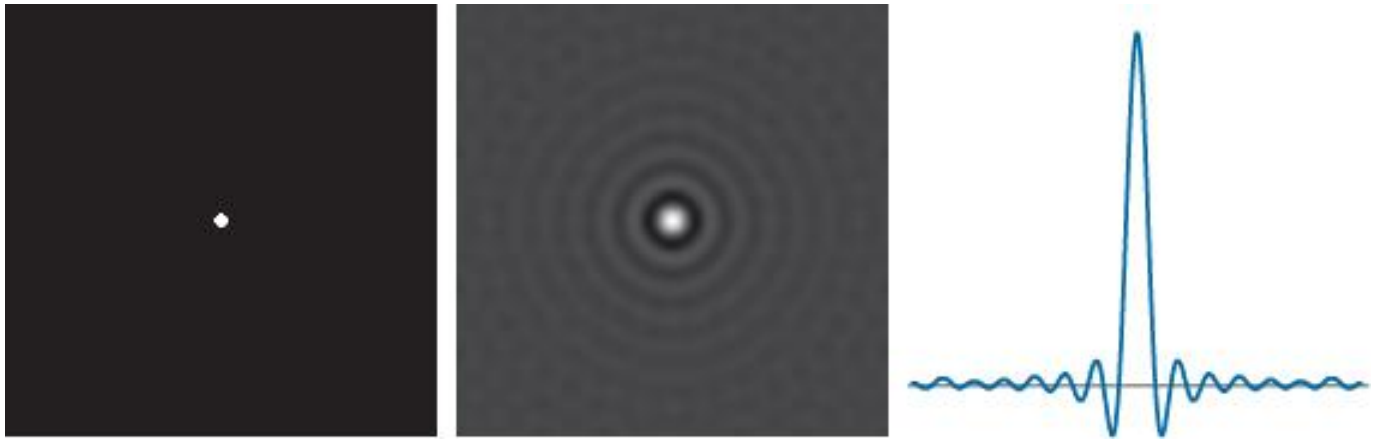
$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{otherwise} \end{cases}, \text{ where } D(u,v) = \sqrt{\left(\left(u - \frac{P}{2}\right)^2 + \left(v - \frac{Q}{2}\right)^2\right)}$$

Distance from center of image

More accurately, use coordinates of $H(0,0)$ **after** centering

Filtering in the frequency domain

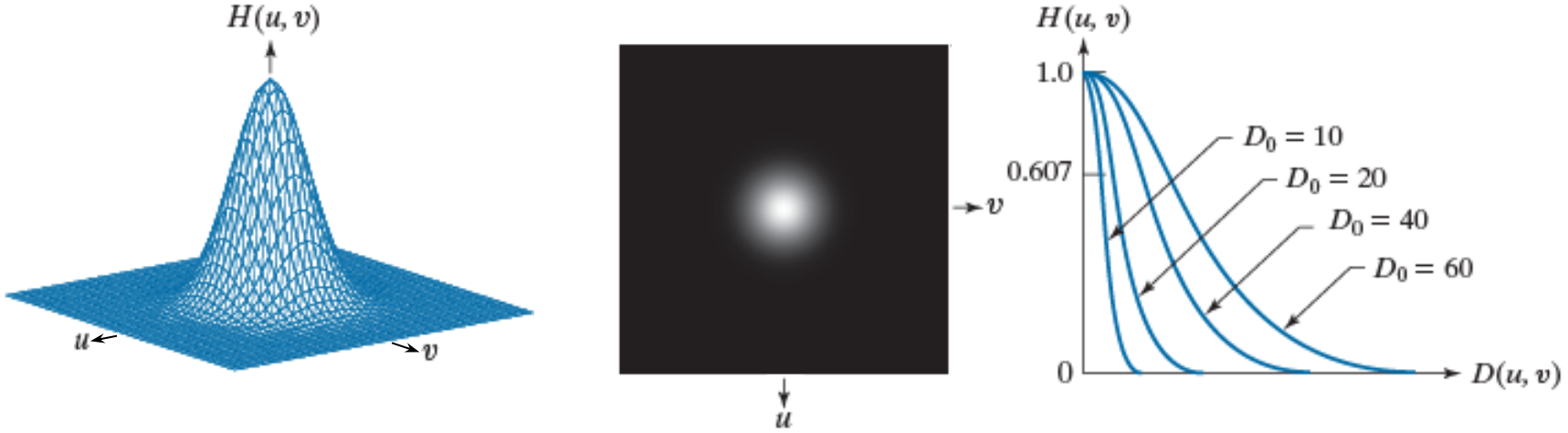
- Ideal lowpass filter (LPF)
  - Frequency domain vs spatial domain



$H(u,v)$ $h(x,y)$

Filtering in the frequency domain

- Gaussian lowpass filter (LPF)
  - Filter can be constructed in frequency domain directly

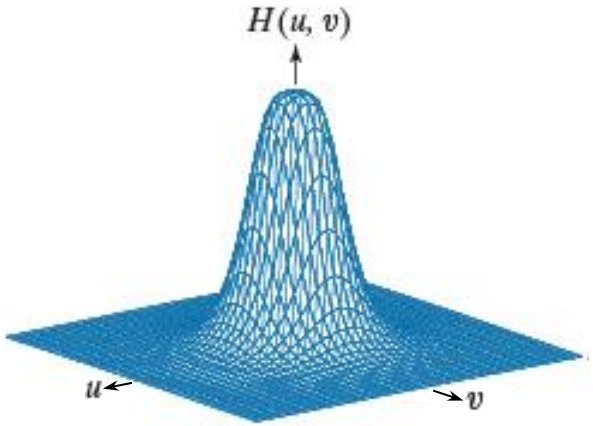


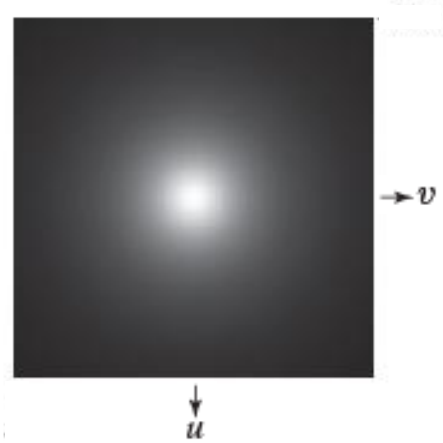
$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}, \text{ where } D(u,v) \text{ is the distance from the center}$$

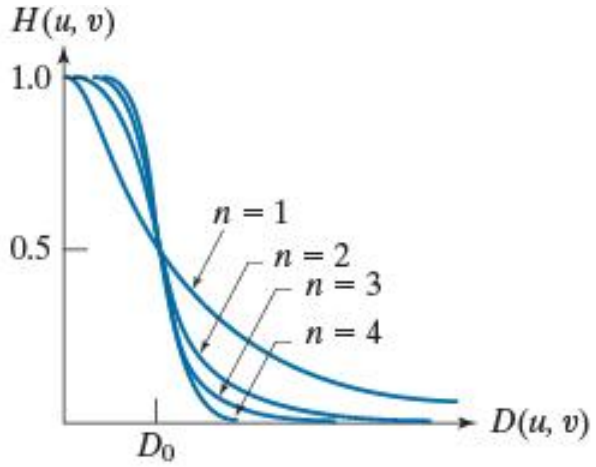
$$\text{Letting } \sigma = D_0,\ H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

Filtering in the frequency domain

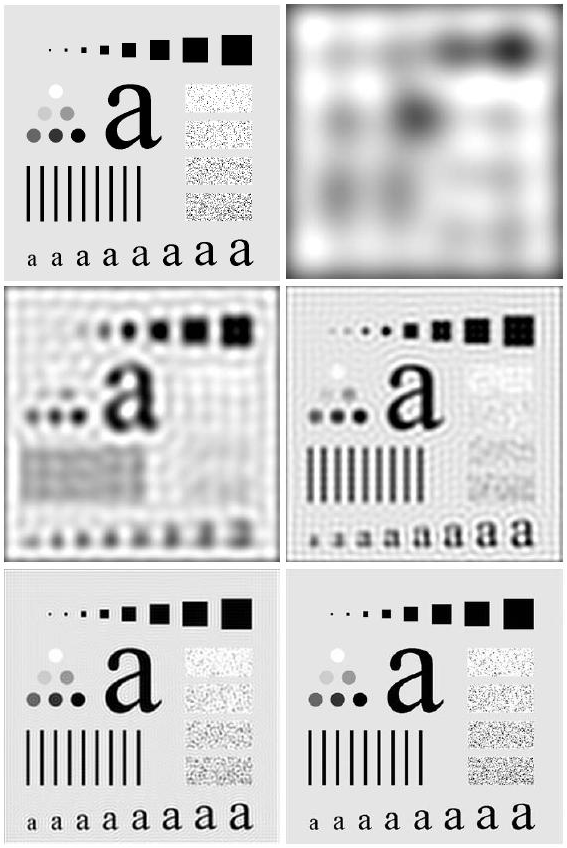
- Butterworth lowpass filter (LPF)



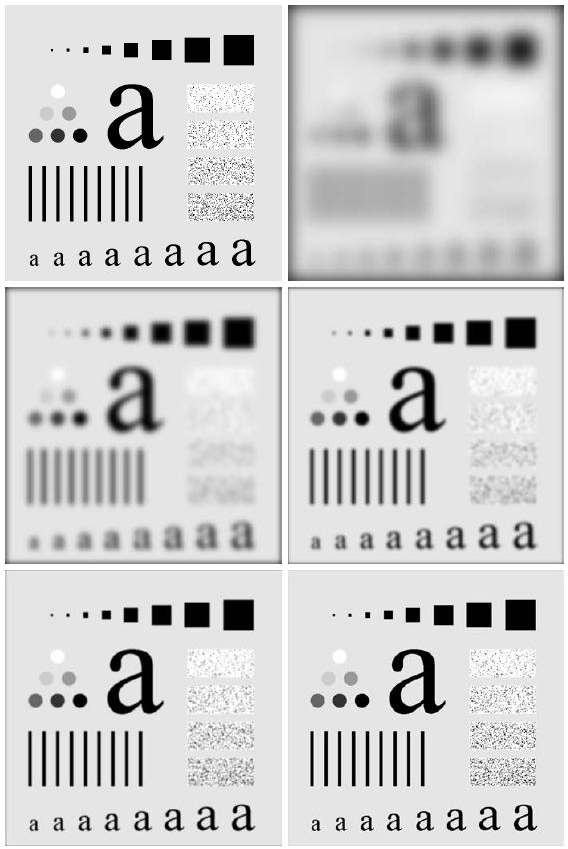




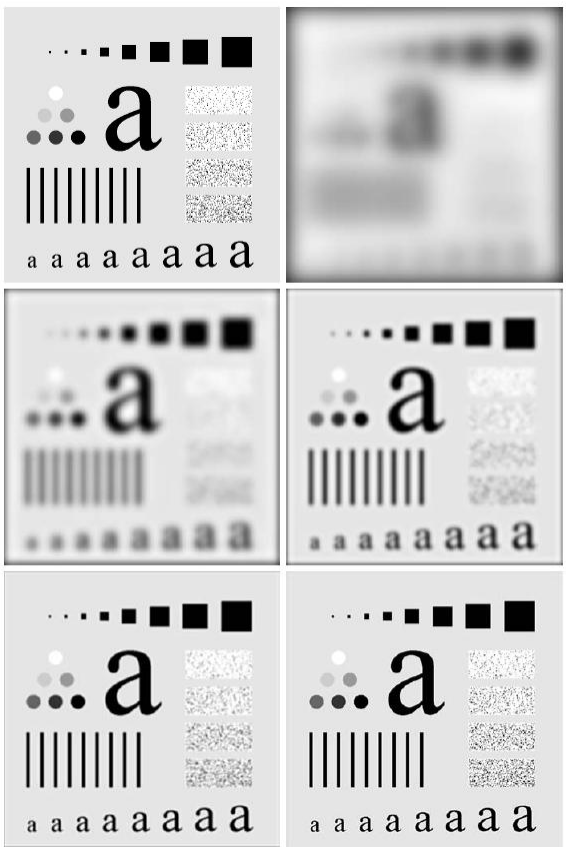
Filtering in the frequency domain



Ideal LPF

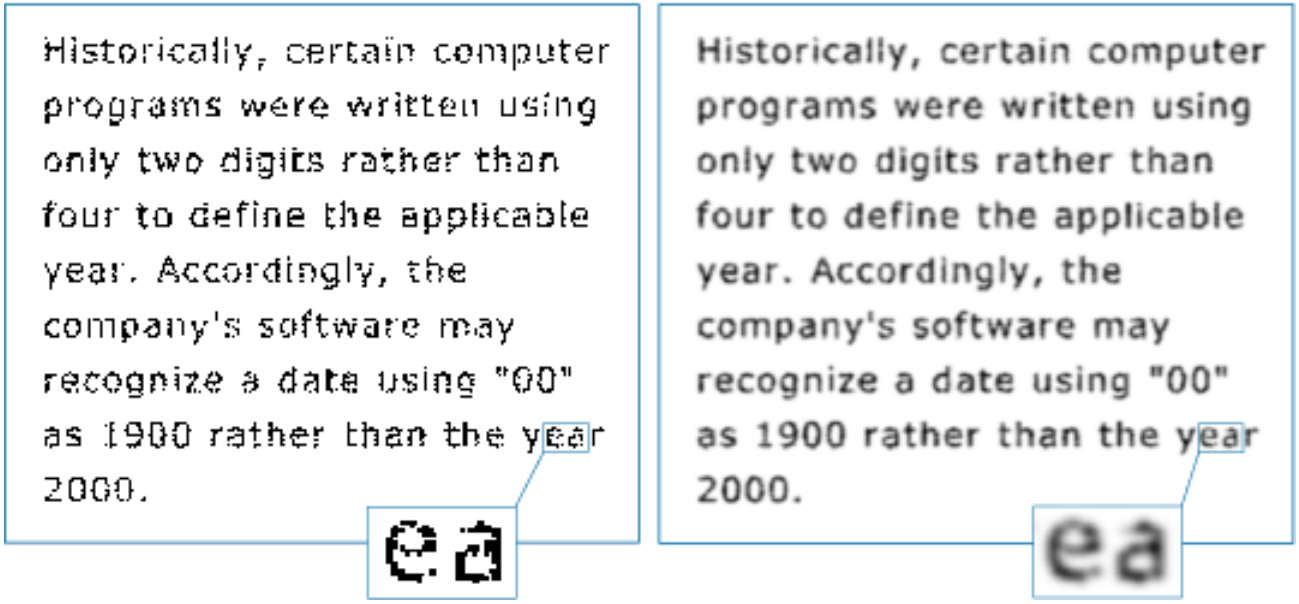


Gaussian LPF



Butterworth LPF

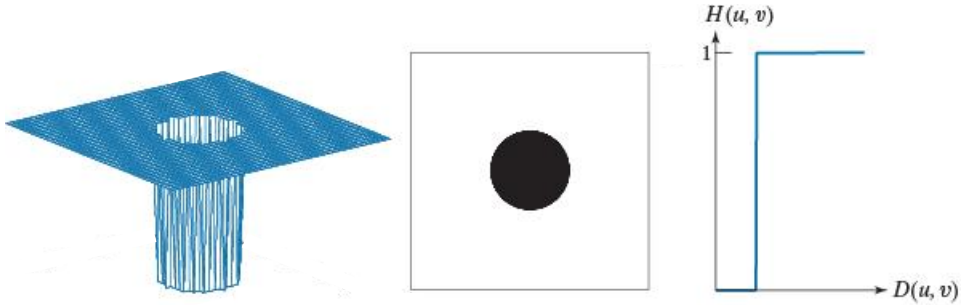
Example: character recognition



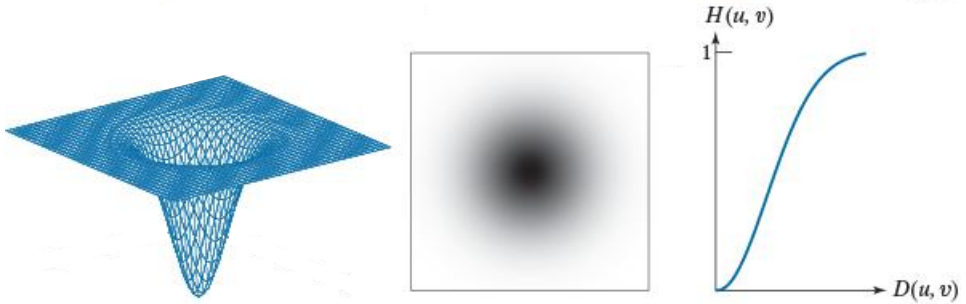
Gaussian LPF
joins broken characters

Highpass filter (HPF) Frequency domain

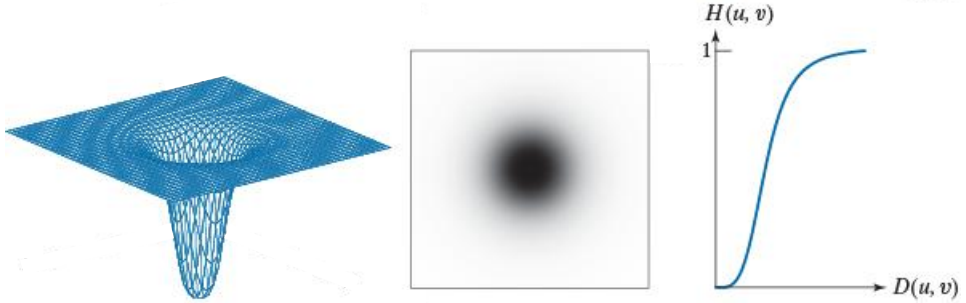
Ideal HPF



Gaussian HPF

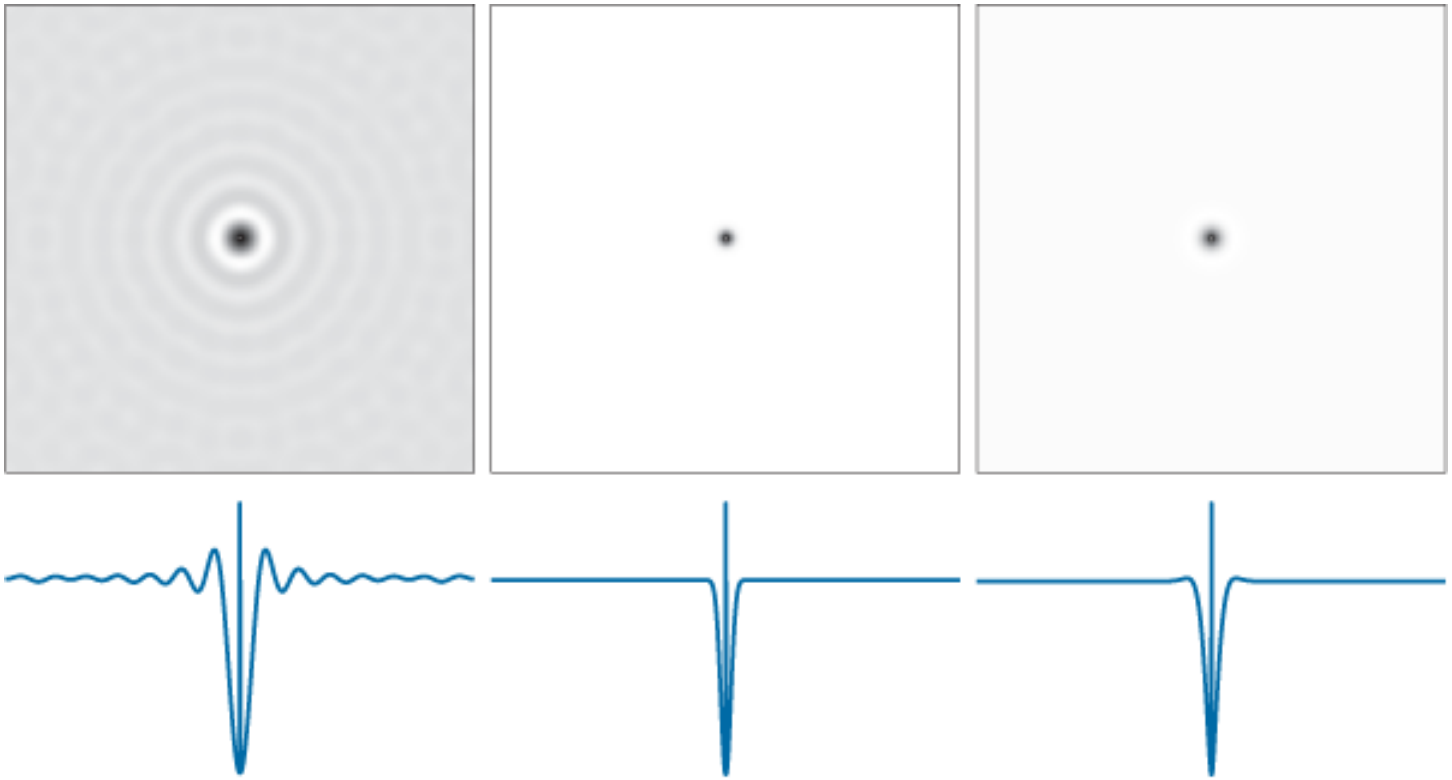


Butterworth HPF



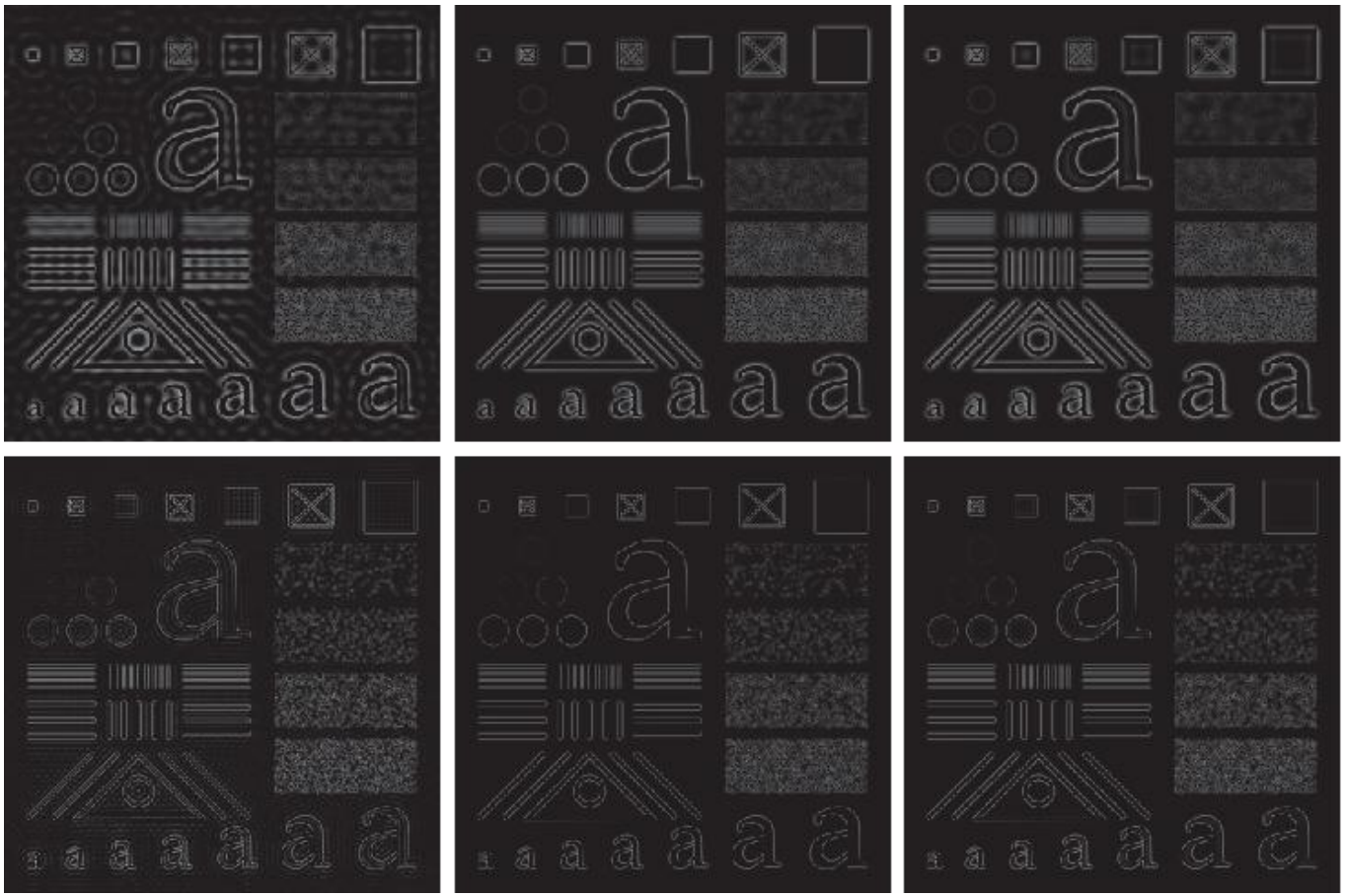
Highpass filter (HPF)
Spatial domain

Ideal HPF | Gaussian HPF | Butterworth HPF

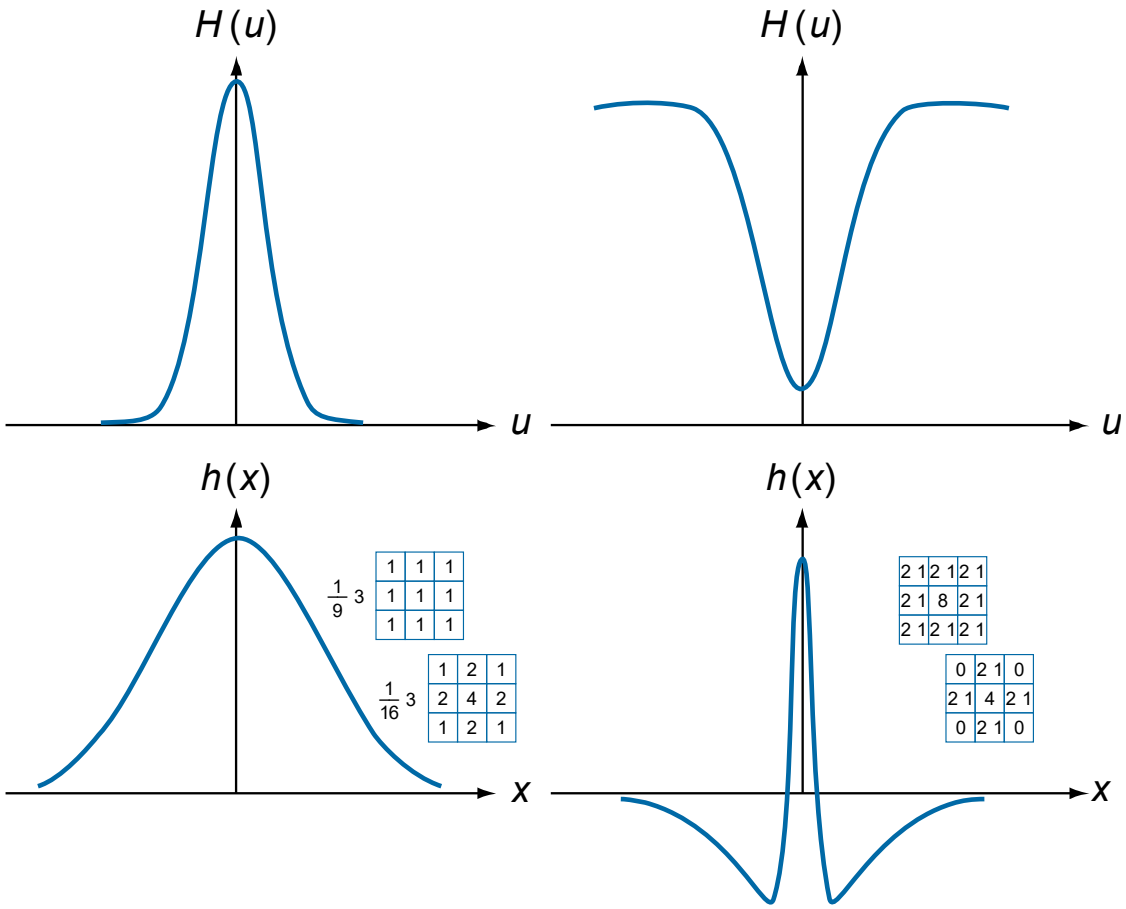


Filtering in the frequency domain

Ideal HPF | Gaussian HPF | Butterworth HPF



Filtering in the frequency domain



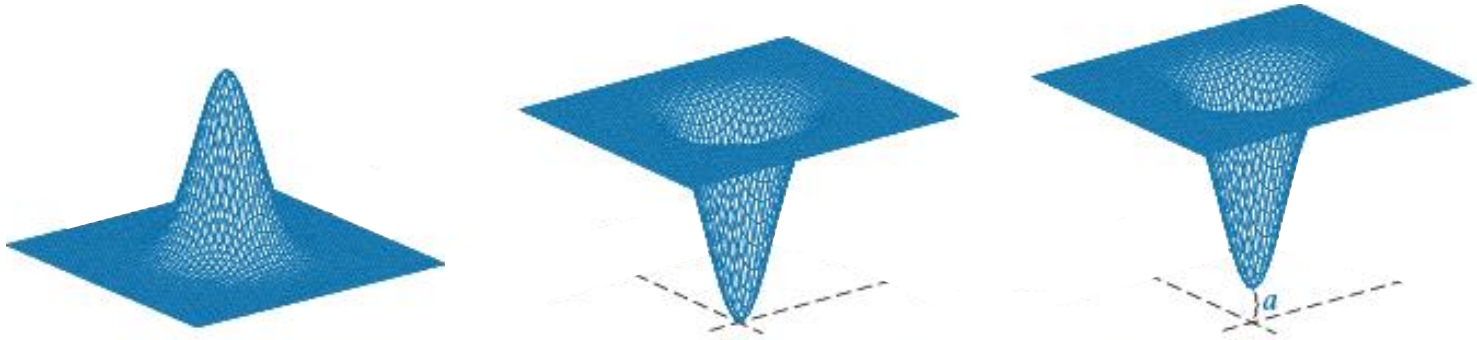
Filtering in the frequency domain

- Sharpening filter



Filtering in the frequency domain

Lowpass filter | Highpass filter | Offset highpass filter

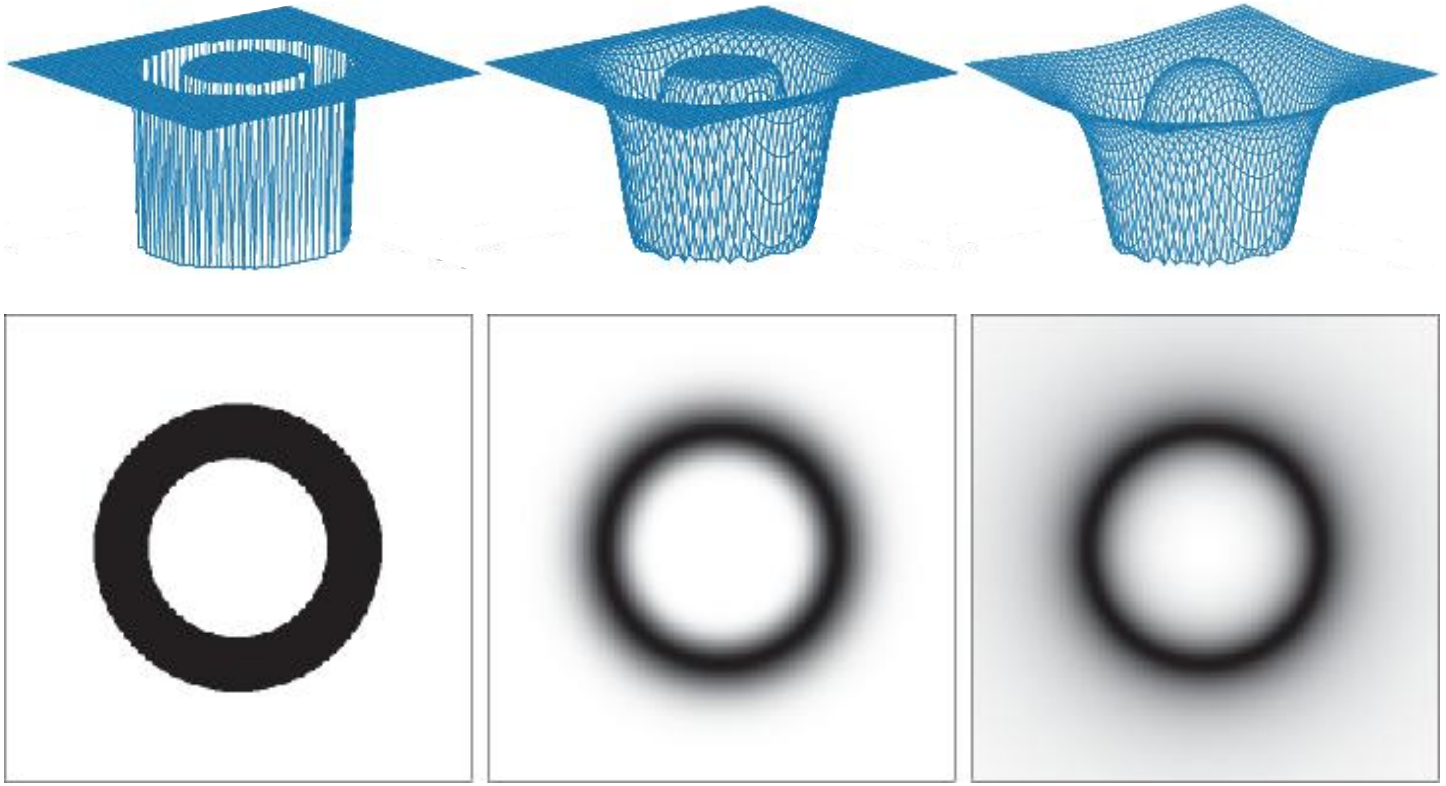


2D

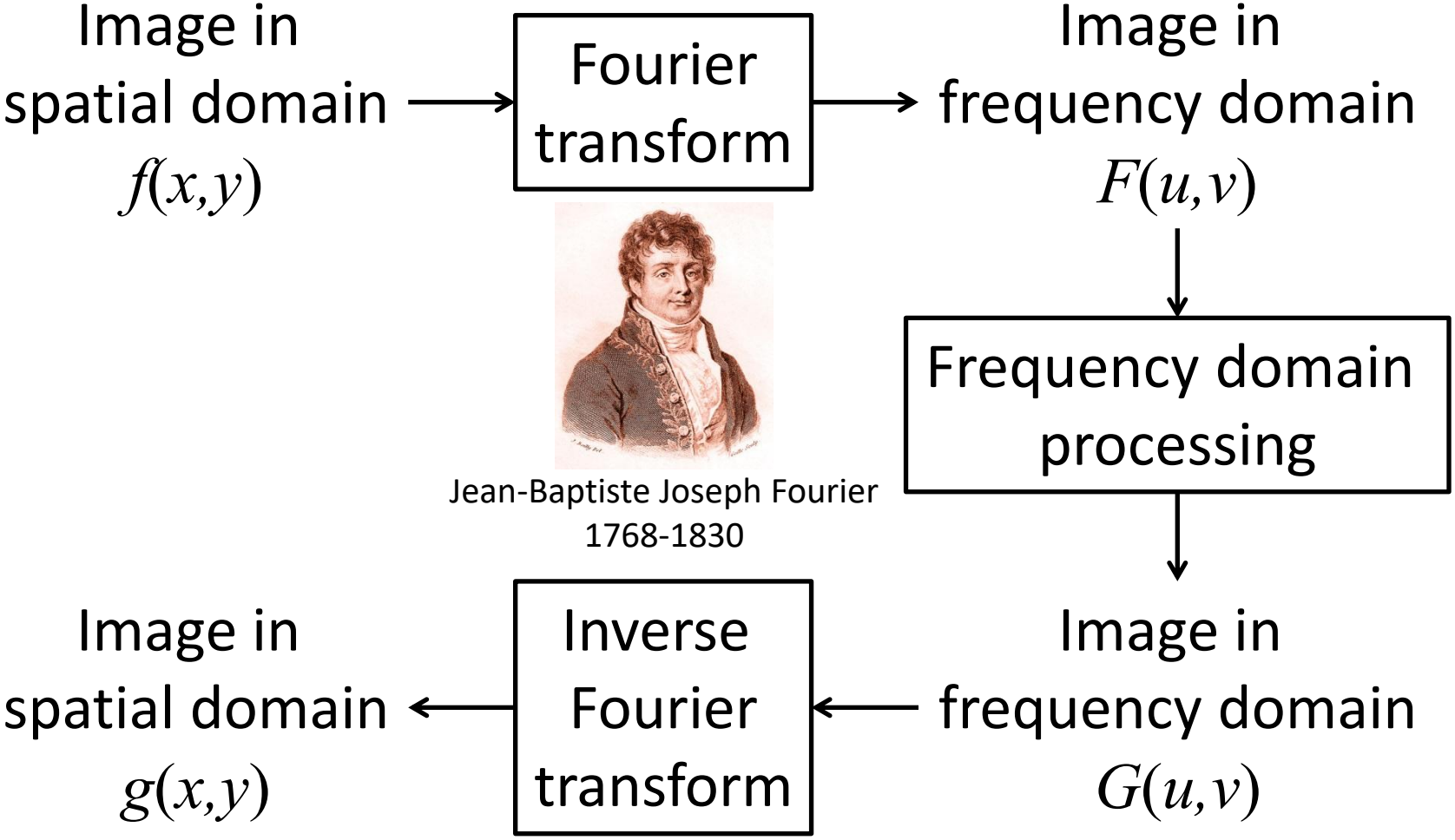


Bandreject filters

Ideal Gaussian Butterworth



Overview: Image processing in the frequency domain



Jean-Baptiste Joseph Fourier
1768-1830

Next Lecture

- Image restoration
- Reading
  - Chapter 5: Image Restoration and Reconstruction
    - Sections 5.1, 5.2, 5.3, 5.4, 5.6, and 5.7